Carl B. Allendoerfer Awards

Juan Arias de Reyna, David Clark, and Noam Elkies

"A modern solution to the Gion Shrine Problem," *Mathematics Magazine*, 92(2), 110–122.
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This article shines a welcome light on a little-known nook of mathematical history: remarkable computationally-intensive geometry problems inscribed in wooden tablets and hung in 18th-century Buddhist temples and Shinto shrines. These formed a part of a cultural flourishing in Japan under the Tokugawa shogunate, in which kabuki theater, haiku poetry, ukiyo-e woodblock printing, and a unique style of mathematics called *wasan* enjoyed a surge of interest. *Wasan* focused on geometric problems, such as packings with circles and polygons. An especially appealing set of theorems would be inscribed on a tablet and hung in a shrine, "as an offering to the gods, a challenge to other worshippers, and an advertisement for the school producing the work," in the words of the authors. Being a closed society, the Japanese drew inspiration from Greek geometry and Chinese computation, and *wasan* problems often blended the two.

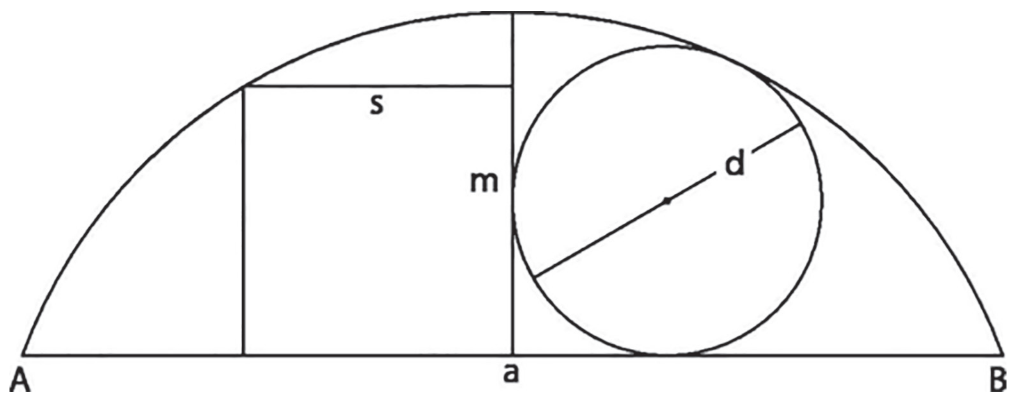

Figure 1: The Gion shrine problem

One particularly famous tablet was found in Kyoto's Gion shrine. It consists of a square and circle inscribed in a circular arc as in Figure 1. The square has side length s, the circle has diameter d, and two other segments relating to the arc have lengths a and m. The challenge is to recover a, d, s, and m given only the quantities $p = a + d + s + m$ and $q = (m/a) + (d/m) + (s/d)$, using the geometrical relationships of the quantities involved.

The first recorded solution was due to Tsuda Nobuhisa in 1749, who gave the desired quantities in terms of the roots of a degree-1024 polynomial. Further 18th century solutions reduced the degree of the necessary polynomial to 46, and then to 10. The latter uses only the Pythagorean theorem, linear algebra, and "a great deal of algebraic persistence," as the authors put it. The authors then give a modern solution that relies on trigonometric functions to describe a, d, s, and m in terms of a simpler degree-10 polynomial, with the added benefits of elucidating the existence and uniqueness of solutions.

The article then turns to a problem that would not have been on the minds of wasan practitioners, but would interest anyone who cares about Diophantine equations: is there a solution to the Gion shine problem where a, d, m, and s are all rational numbers? This leads to the problem of finding all solutions to $y^2 = x^3 - x$ with both x and y rational. Going back at least to Fibonacci, this problem was solved by Fermat using his famous method of descent, which has reverberated through modern number theory.

The article concludes with remarks about some more recent developments in the study of Diophantine equations. Thus a voyage that started in centuries-ago Japan winds up near the present day. The article gives a vivid portrait of how widely separated parts of mathematical history can in fact be intertwined, and reminds us of the universality of mathematics in the human experience.

Response

It is a pleasure and a surprise to be awarded the Allendoerfer prize this year. In addition to being an interesting and challenging exercise, the Gion Shrine problem gave us an opportunity to portray mathematics at the intersection of art, recreation, science, history, and culture. When we arrived at the solution, one of us (Arias de Reyna, who lived under a dictator in Spain and did not have access to math books) noticed how he would have enjoyed reading it as a teenager. Our paper is written, in large part, for this young person. We hope that our solution—weaving together geometry, trigonometry, and algebra—might be read and enjoyed by someone with a similar passion for mathematics. The last part of the paper on Diophantine equations was a relatively late but very welcome addition, revealing the Gion shrine problem to be another point of intersection, this time between traditional Japanese math and Western number theory.

Biographical Sketches

J. Arias De Reyna learned mathematics from books starting at age thirteen; at the time, even books were difficult to get in dictator Franco's Spain. He has published a book about Carleson's proof on the convergence of Fourier series, defining the largest known rearrangement invariant space of functions with almost everywhere convergent series. He also obtained good bounds for the Riemann-Siegel expansion.

David Clark was trained as a quantum topologist, but has recently become interested in the history of Japanese mathematics. In 2017 he hosted an international conference on the topic in Ashland, Virginia. Clark regularly takes students to Japan to learn about sangaku tablets, and has written about his experiences in Math Horizons.

Noam D. Elkies is a number theorist, much of whose work concerns Diophantine geometry and computational number theory. He was granted tenure at Harvard at age 26, the youngest in the University's history. Outside of math, Elkies' main interests are music—mainly classical piano and composition—and chess, where he specializes in composing and solving problems.