George Pólya Awards

The George Pólya Awards, established in 1976, are made to authors of articles of expository excellence published in the *College Mathematics Journal*. The awards are named for George Pólya, who was a distinguished mathematician, well-known author, and professor at Stanford University.

J.M. Christian and H.A.J. Middleton-Spencer

"On the *N*th Roots of –1 and Complex Basin Boundaries: Fractals from Newton-Raphson," *The College Mathematics Journal*, 51:2, 95–104. 10.1080/07468342.2020.1703452

Beginning with the simple question "Can you find the roots of a quadratic equation?", the reader is led seamlessly from quadratic equations to cubic equations to the *N*th roots of –1—creating beautiful and enchanting color pictures along the way. Of course, the roots of –1 are well-known. But, this article focuses on the process of finding them numerically through the Newton-Raphson method. Since the results of that iterative scheme involve the choice of an initial input, the authors color-code regions of the Argand plane corresponding to whether or not Newton-Raphson leads to convergence or non-convergence.

With their exposition, Christian and Middleton-Spencer convey a sense of surprise and wonder as they take an elementary numerical scheme for root-finding and guide the reader to the unexpected emergence of stunning fractal patterns. The origins of those patterns can be viewed through the prism of dynamical systems, with some terminology introduced along the way. Readers are drawn into this amazing universe, invited to explore the alluring pull of the attractors, and see the mind-bending Wada property. The underlying geometry of the basins of attraction for these roots is identified and the reader is led carefully from one exploration to the next by rotations in the Argand plane. Finally, the authors illustrate how the uncertainty dimension can be estimated in order to quantify the degree of complexity in fractal basin boundaries. Such a property has very deep implications in physics and mathematics, and it is linked inextricably to the famous butterfly effect.

The problem is stated straightforwardly, yet challenges the reader to consider larger questions of nonlinearity, chaos, and fractals. The explorations presented in this paper can engage undergraduates and established mathematicians alike in some interesting mathematics, physics, and computation in a way that is both simple and elegant. All readers will enjoy this exciting journey.

Response

Our paper started life as a short piece of coursework undertaken by Holly during James's computational physics classes in 2016/17. From there it snowballed into a collaborative venture, as one question after another arose to confound us both. Over a year later, we were able to answer some of those questions (at least to our own satisfaction) and the decision to try and publish was a no-brainer. We were hoping to entice students into playing with Newton-Raphson fractals for themselves, discovering the beauty and complexity hidden in such deceptively simple maps. Moreover, we wanted them to have as much fun in so-doing as we'd had. *The College Mathematics Journal* (*CMJ*), with its relaxed and informative style, was the perfect vehicle: another no-brainer.

We are profoundly grateful to *CMJ* for seeing fit to publish our work in the first place, and to the Mathematical Association of America for bestowing the George Pólya Award upon us. It is indeed an honor and a privilege to have our paper selected in the midst of so many superb articles that inspire and enthuse the mathematics students of the day. We hope we have given them something new to think about, and we thank *CMJ* for the opportunity to be part of its readership's educational experience.

At their heart, *CMJ* and the George Pólya Awards celebrate education. As such, JMC wishes to acknowledge the historical role played by Dr. Richard J. Potton in providing part of the motive power behind of our article. I was introduced to the fascinating field of nonlinearity and chaos—including the classic cube roots of –1 problem—as a student sitting in Richard's inspirational lectures some 20 years ago. His teaching instilled in that ignorant young undergraduate a love of physics and mathematics which survives intact to this day.

Biographical Sketches

James Christian has spent his entire academic career at the University of Salford, UK. He received his undergraduate (MPhys) degree in 2002, by which time it had become apparent that experiments were not for him. Rather than endure more laboratory work, he undertook a PhD in Theoretical Physics—focusing mainly on solitons in nonlinear optics—which was awarded in 2006. He spent the next five years working as a postdoc, and was for some reason appointed Lecturer (Assistant Professor) in 2011. Nowadays, his research interests include electromagnetic scattering problems (particularly those involving fractal screens), spontaneous pattern formation and, most recently, boundary-integral formulations in fluid dynamics. Teaching-wise, over the years he has lectured on relativity, statistical physics, nonlinear phenomena, photonics, classical waves, mechanics, and vector calculus.

Holly Middleton-Spencer started her PhD in Applied Mathematics at Newcastle University in 2018. She works on modelling Bose-Einstein Condensates with particular focus on how turbulence is generated in experiments and the visualization of said condensates. She completed her bachelor's and master's degrees at the University of Salford, under James Christian. When not doing mathematics, she spends her time playing the flute, trying to master various languages and reading about ancient mathematics.