George Pólya Awards

Adrian Rice and Ezra Brown

"Why Hamilton Couldn't Multiply Triples," *The College Mathematics Journal*, 52:3, 185–192.
doi.org/10.1080/07468342.2021.1897418

The editorial guidelines for *The College Mathematics Journal* state that it "is designed to enhance classroom learning and stimulate thinking regarding undergraduate mathematics" and is "aimed at the college mathematics curriculum with emphasis on topics taught in the first two years." Adrian Rice and Ezra Brown's article "Why Hamilton Couldn't Multiply Triples" provides an excellent exposition within these guidelines of part of the history and properties of quaternions. Rice and Brown guide their readers through the development of quaternions, using some interesting mathematics including college algebra, number theory and linear algebra, and providing just enough background in each case to make this an interesting and accessible article. It could provide the seed for student projects in many undergraduate mathematics courses, from history to number theory. The exposition is smooth, engaging, and easily captures the interest of the reader.

Mention the name Hamilton to a colleague, and the reply probably involves quaternions. The Irish mathematician Sir William Rowan Hamilton developed these "numbers" in a search to extend complex numbers, modeled in two-space, into something requiring three-space. Logically, one might simply just extend complex numbers with one more term, $z = a + bi + cj$, where $i^2 = j^2 = -1$. The operations of addition and subtraction are naturally defined, yet multiplication creates a stumbling block. For example, what is ij? If $ij = \pm 1$, then $iij = \pm i$, which implies that $j = \pm i$, which is not helpful. As the authors put it, "Hamilton and his contemporaries quickly found that they could not multiply two triples together to form another triple—the multiplication just didn't work."

As the authors note, the story of Hamilton's discovery of quaternions is well known, including finding that he really needed to move beyond triples and work with a four-termed expression: $a + bi + cj + dk$, where a, b, c, and d are real numbers and $i^2 = j^2 = k^2 = -1$.

The popularity of quaternions diminished as vectors became popular, but has seen a resurgence in, as the authors note, applications in physics, engineering, and computing. In computer graphics, using quaternions instead of vectors to create an image avoids certain problems that render an image improperly.

Professors Rice and Brown focus on why triples proved so frustrating to Hamilton. They pose the question "why was Hamilton unable to create a coherent system of algebraic triples in the first place." Their approach: "In modern terminology, Hamilton and his contemporary mathematicians were trying to find a normed algebra over the real numbers." One of the strengths of the authors' exposition is that this precedes a nice description of some properties of a normed algebra and proceeds to use them, along with Hamilton's famous brainstorm that the products of these imaginary terms were not commutative (famously, while crossing the Brougham Bridge in Dublin) coupled with some number theory, to show that the search for triples was hopeless. Additionally, Rice and Brown point out that Euler (of course!) had essentially discovered the properties of quaternions 95 years before Hamilton.

The authors follow this with a well-written "quick linear algebra review," looking at characteristic polynomials and eigenvalues, to provide the tools needed for their "You can't multiply triples theorem."

The smooth exposition, while accessibly combining several areas of undergraduate mathematics applied to an interesting topic, makes this useful article a model for articles for this journal.

Response

We are thrilled, honored, and grateful to the Pólya Committee for this award. We thank the *College Mathematics Journal* editor Dominic Klyve and his staff of referees for accepting our paper and improving it with their comments and suggestions. This paper stemmed from our joint interest in the history of mathematics, especially in the mid-nineteenth century events that led up to the discoveries of the quaternions and the octonions. In particular, we wondered about Hamilton's troubles with triples, and the paper being honored was the result. It was also fun to write! It was at a meeting of the MAA's Maryland/DC/Virginia Section where we first met, another such meeting where we first presented this paper, an MAA journal that published the paper, and an MAA committee that honored our collaboration. So, four-fold thanks to the MAA!

Biographical Sketches

Adrian Rice is the Dorothy and Muscoe Garnett Professor of Mathematics at Randolph-Macon College in Ashland, Virginia, where his research focuses on nineteenth- and early twentieth-century mathematics. In addition to papers on various aspects of the history of mathematics, his books include *Mathematics Unbound: The Evolution of an International Mathematical Research Community, 1800–1945* (with Karen Hunger Parshall), *Mathematics in Victorian Britain* (with Raymond Flood and Robin Wilson), and *Ada Lovelace: The Making of a Computer Scientist* (with Christopher Hollings and Ursula Martin). In his spare time, he enjoys music, travel, and spending time with his wife and son.

Ezra (Bud) Brown grew up in New Orleans, has degrees from Rice and LSU, taught at Virginia Tech for 48 years, and retired in 2017 as Alumni Distinguished Professor Emeritus of Mathematics. He does research in number theory, combinatorics, and the history of mathematics. He enjoys finding and writing about connections between seemingly unrelated mathematical topics, an interest stemming from graduate school that has never left him. He and the late Richard Guy are the authors of the Carus Monograph *The Unity of Combinatorics*, which was published in May 2020.

Bud enjoys baking biscuits, singing (anything from opera to rock-n-roll), playing jazz piano, watching birds, and—since his retirement—traveling with his wife Jo. His favorite number is 265.