Complex Analysis

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Complex analysis is one of the most beautiful as well as useful branches of mathematics.
--Murray R. Spiegel.

Introduction: Complex analysis is indeed a beautiful and useful branch of mathematics. It is one of the classical subjects with most of the main results extending back into the nineteenth century and earlier. Yet, the subject is far from dormant. It is a launching point for many areas of research and it continues to find new areas of applicability, from pure mathematics to applied physics. Many of the giants of mathematics have contributed to the development of complex analysis. Names such as Euler, Cauchy, Gauss, Riemann, and Weierstrass are common occurrences among its list of important results. Technology has opened additional avenues of study using complex analysis, from fractals to color-enhanced methods for visualization. Complex analysis is an important component of the mathematical landscape, unifying many topics from the standard undergraduate curriculum. It can serve as an effective capstone course for the mathematics major and as a stepping stone to independent research or to the pursuit of higher mathematics in graduate school.

In preparing this report, it quickly became apparent that providing guidelines for an undergraduate course in complex analysis would be an exercise in compromise. The subject, its range of applications, and the multitude of additional resources that can be employed is vast. The content of a particular course and the pedagogical methods used depend on many factors: the background of the students, the technological resources that are available, the textbook chosen, the interests of the students, the role of the course within the program of study, and the whims of the instructor, to name just a few. Despite the wide range of implementations, there are some common threads that should permeate most offerings of a Complex Analysis course. Here, we present what we see as the core material, content that forms the cornerstone of the subject. Even within this core, however, there is much room for interpretation and emphasis. Beyond the core curriculum, we offer some suggestions for various extensions that may appeal to different instructors or groups of students.

Student Audience: Many mathematics departments offer a one-semester course in undergraduate Complex Analysis. The course is typically offered at the junior or senior level, though it may also be offered as a dual enrollment with a first year graduate-level course. While the course is taken primarily by mathematics majors, it is also applicable to students in physics and engineering. For this reason, it is typically taught without a strong emphasis on proofs.

Complex Analysis is rarely a requirement for completing the mathematics major. It is more often an elective or perhaps a required course within a particular sub-track of the major. The prerequisites for the course vary widely, but multivariable calculus seems to be the standard minimum requirement among most programs. Linear algebra is often another prerequisite, but is by no means universal. Other possible prerequisites include an introduction to proofs course, differential equations, and real analysis. The background of the students directly affects the type of course that can be taught. Fewer mathematical prerequisites allow more students from other majors to take the course. Complex Analysis is particularly well-suited to physics majors. It was noted that all “serious physics majors” should take Complex Analysis.

The course is also very useful for students planning to go to graduate school in mathematics or applied mathematics. Many graduate programs offer a qualifying exam in real and complex analysis. A solid undergraduate experience can ease the transition to graduate courses and set the stage for more advanced study in a number of interesting and current fields. Students planning to do undergraduate research or a senior capstone may also find a course in Complex Analysis as a useful stepping stone to projects in complex dynamics, differential geometry, or analytic number theory.

Complex analysis is a subject that can serve many roles for different majors and types of students. The material and theorems reach into many areas of pure and applied mathematics. At one end of the spectrum, the subject provides a powerful set of tools for dealing with the theory of integration, linear differential equations, and infinite series, while at the other end it is a deep and unifying subject that has a strong pure mathematics feel. For these reasons, the course cuts across traditional niche groups and instead can be a mathematically rich experience for pure and applied mathematics majors, as well as students in engineering and physics.

Cognitive Goals: A course in Complex Analysis has the potential to address many learning outcomes that are important in studying mathematics. The subject has connections to several other mathematical areas and it provides students with opportunities to build a deeper cognitive mathematical framework.

Many results and concepts in the field can be presented both analytically and geometrically. This is true from the basic arithmetic of complex numbers, to the local behavior of analytic functions, to more advanced results such as the argument principle or the maximum modulus principle. Wherever possible, results and problems should be presented both analytically and geometrically. Often, the more subtle ideas in the subject are easily understood from a geometric perspective.

Another key feature of Complex Analysis is the wide range of applications that can be used in a first course. Complex analysis is inherently two dimensional—in some sense it is calculus in the plane. Thus, the subject lends itself to problems that are naturally defined in the plane such as electric fields, ideal fluid flow, fields with point sources or sinks, and many other examples. Complex integration, a central feature of any course in the subject, has deep connections to real line integrals in the plane.

In developing the material for students, it is easy to motivate the general results through examples. Students can often “discover” fundamental ideas through a few carefully chosen cases. Properties and theorems such as how multiplication of complex numbers affects their arguments, Cauchy’s Theorem, or formulas for computing residues follow naturally after working through examples. The entire subject is well-suited to working from the particular to the more abstract.

In terms of developing a student's mathematical thinking and understanding, complex analysis provides a framework for connecting many ideas from other parts of the standard undergraduate mathematics curriculum. Through a study of complex analysis, previously disjoint topics such as differentiation, integration, power series, and vector fields are brought under one unifying theory. Moreover, the standard functions from calculus and engineering (exponential, logarithmic, trigonometric, hyperbolic trigonometric, polynomials, and rational) are seen to have much more in common than most students might have realized. On their natural domains in the complex plane, properties such as periodicity, boundedness, location of zeros, and mapping properties become more natural and consistent. Students realize that their previous view of these functions was merely seeing a slice and not the full picture. In the complex plane, notions of branch cuts and branch points become fundamental and lend a practical importance to carefully defining domains and ranges. Taylor series expansions of functions and their regions of convergence become far less mysterious when viewed in the complex plane. Fundamental ideas and methods from multivariable calculus often make more sense when viewed through the lens of complex function integration theory. Thus, a course in complex analysis naturally unifies many topics in a conceptually consistent way.

A course in complex analysis is the jumping off point for many areas of current research and popular interest. Topics such as fractals, complex dynamics, minimal surfaces, and harmonic functions are within easy reach. Students with a background in complex variables can also study the Riemann-zeta function and begin to appreciate the Riemann hypothesis and its connection to the prime numbers.

Finally, a course in complex analysis provides a wonderful playground for experimenting with technology. Modern computer algebra systems and many freely available applets allow amazing visualization of complex functions and their properties. Complex functions, when viewed as real mappings, have graphs that live in four-dimensional space. Thus, viewing complex functions requires other tools such as using color to plot the argument, while a standard surface depicts the modulus. Color-enhanced plots of complex functions bring an entirely new way of "seeing" the graphs of complex functions. In addition, technology allows students to easily experiment with the mapping properties of a complex function or allows them to view it as a vector field. Having multiple representations helps students become more comfortable in understanding and working with complex functions.

Technology fits naturally in a Complex Analysis course. It can be used to almost any degree desired, from instructor-led demonstrations, to student-generated code for exploring a complex dynamical system. It is possible to build an entire course around a particular CAS platform or, on the other end of the spectrum, to use a few point and click applets to help students better visualize the subject. While it is certainly possible to teach the subject without using any technology, this would miss out on an excellent opportunity to incorporate modern technology in a meaningful and highly accessible way.

Sample courses: We provide a core set of topics that are found in almost all complex analysis texts. These topics form the foundation of the subject and provide the tools needed to explore other topics and applications. We believe that most of the topics listed should be part of any first course in Complex Analysis.

In providing sample syllabi, we have listed the topics in a reasonable order of presentation, though other orders are certainly possible. We have also suggested the number of lessons that should be devoted to each topic. Our working assumptions were that a typical course would contain forty lessons of 50 minutes each. We have tried to make our core set of topics use 25 to 30 lessons, leaving the remaining lessons for additional topics and assessment.

Following our core set of topics, we present suggestions for three additional courses that each give a different emphasis: the first has the core material plus additional emphasis on more pure mathematics topics, the second has the core plus an introduction to complex dynamics, and the third has the core topics plus an applied emphasis. These syllabi represent a sampling of the myriad possibilities in designing a vibrant undergraduate course in Complex Analysis.

Assumed prerequisite: multivariable calculus. We make particular mention of how to use technology, since technology use depends heavily on the IT infrastructure of the institution.

Sample syllabus for core topics, only

Topics	Approx. # of lessons
Complex numbers, complex arithmetic, geometric representation, polar and exponential representation, modulus, argument, Euler's identity, DeMoivre's formula, roots of unity, basic topology of sets in the plane (open, closed, connected, bounded, etc.).	3 - 4
Complex functions to include multiple-valued functions and the notion of branches. The geometry of complex functions as mappings from the z-plane to the w-plane.	3
The typical functions from calculus extended to the complex domain (polynomials, power functions, rational, exponential, logarithmic, trigonometric, and hyperbolic trigonometric).	3 - 4
The theory of differentiation for complex functions. Analytic functions to include the Cauchy-Riemann equations, harmonic functions, and properties of analytic functions.	3 - 4
The theory of integration of complex functions to include the Cauchy-Goursat theorem, Cauchy's Integral formula, and several consequences of the Integral Formula to include Cauchy's Inequality, Liouville's theorem, and the Maximum Modulus Principle.	5 - 6
The theory of Taylor and Laurent series. Constructing series that converge in certain circular or annular regions.	3 - 4
Classification of zeros and poles.	2
Definition and computation of residues. Using residues to compute integrals around closed loops in multiply-connected domains	3

Sample Syllabus for Core Topics, plus an additional emphasis in pure mathematics

Topics	Approx. # of lessons
Complex numbers, complex arithmetic, geometric representation, polar and exponential representation, modulus, argument, Euler’s identity, DeMoivre’s formula, roots of unity, basic topology of sets in the plane (open, closed, connected, bounded, etc.).	3 - 4
Complex functions to include multiple-valued functions and the notion of branches. The geometry of complex functions as mappings from the *z*-plane to the *w*-plane.	3
The typical functions from calculus extended to the complex domain (polynomials, power functions, rational, exponential, logarithmic, trigonometric, and hyperbolic trigonometric).	3 - 4
The theory of differentiation for complex functions. Analytic functions to include the Cauchy-Riemann equations, harmonic functions, and properties of analytic functions.	3 - 4
The theory of integration of complex functions to include the Cauchy-Goursat theorem, Cauchy's Integral formula, and several consequences of the Integral Formula to include Cauchy's Inequality, Liouville's theorem, and the Maximum Modulus Principle.	5 - 6
[A] *Converses of Cauchy’s theorem: Morera’s theorem, Schwartz reflection principle*	*2*
The theory of Taylor and Laurent series. Constructing series that converge in certain circular or annular regions.	4
Classification of zeros and poles.	2
[B] *Infinite products, gamma & zeta function*	*2*
Definition and computation of residues. Using residues to compute integrals around closed loops in multiply-connected domains.	3
[C] *Applications of the residue theorem: infinite sums, generating function computations*	*2*
[D] *Conformal mappings and the Riemann mapping theorem*	*4*
[E] *Poisson formulas and the Dirichlet problem*	*4*

The topics not in italics are considered essential. [A]—[E] are optional topics, independent of each other; their places in the above list suggest their placing in the curriculum. If the optional topics should add up to 10 hours, the possibilities are

- [A], [B], [C], and one of [D] and [E]
- [D], [E], and one of [A], [B], and [C].

Sample Syllabus for Core Topics plus an additional emphasis in Complex Dynamics

Topics	Approx. # of lessons
Complex numbers, complex arithmetic, geometric representation, polar and exponential representation, modulus, argument, Euler's identity, DeMoivre's formula, roots of unity, basic topology of sets in the plane (open, closed, connected, bounded, etc.). *Complex iteration, fixed points, periodic points, pre-periodic points.*	5
Complex functions to include multiple-valued functions and the notion of branches. The geometry of complex functions as mappings from the z-plane to the w-plane.	3
The typical functions from calculus extended to the complex domain (polynomials, power functions, rational, exponential, logarithmic, trigonometric, and hyperbolic trigonometric).	4
The theory of differentiation for complex functions. Analytic functions to include the Cauchy-Riemann equations, harmonic functions, and properties of analytic functions.	4
Convergence to a complex number, convergence to infinity, bounded sequences, basin of attraction, classification of fixed points and cycles (attracting, repelling, neutral) based on convergence and on the modulus of the derivative, critical orbits, open and closed sets, Cantor sets, connected sets.	4
Filled Julia sets for the family $z^2 + c$ *and corresponding Fatou components, properties of Julia sets and Fatou components.*	3
Mandlebrot set and relationships between period bulbs, limbs, antennae, and Fatou components.	2
The theory of integration of complex functions to include the Cauchy-Goursat theorem, Cauchy's Integral formula, and several consequences of the Integral Formula to include Cauchy's Inequality, Liouville's theorem, and the Maximum Modulus Principle.	6
The theory of Taylor and Laurent series. Constructing series that converge in certain circular or annular regions.	4
Classification of zeros and poles.	2
Definition and computation of residues. Using residues to compute integrals around closed loops in multiply-connected domains	3

* If you want even more of a complex dynamics approach, some of the integration theory could be replaced with an analysis of Julia sets related to Newton's method or with more exploration with Julia set and Mandelbrot set applets.

Sample Syllabus for Core Topics, plus an emphasis in applied mathematics

Topics	Approx. # of lessons
Complex numbers, complex arithmetic, geometry of complex numbers in the complex plane. Polar and exponential representation of complex numbers, Modulus, conjugate, and argument of complex numbers (Euler's identity, DeMoivre's formula, roots of unity, etc.).	2
Complex functions to include multiple-valued functions and the notion of branches. The geometry of complex functions as mappings from the z-plane to the w-plane, Möbius transformations.	3
The typical functions from calculus extended to the complex domain (polynomials, power functions, rational, exponential, logarithmic, trigonometric, and hyperbolic trigonometric).	4
The theory of differentiation for complex functions. Analytic functions to include the Cauchy-Riemann equations, harmonic functions, and properties of analytic functions.	2
Flows, Fields, and analytic functions – complex potentials, Dirichlet Problem.	2
The theory of integration of complex functions to include the Cauchy-Goursat theorem, Cauchy's Integral formula, and several consequences of the Integral Formula to include Cauchy's Inequality, Liousville's theorem, and the Maximum Modulus Principle.	4
Vector Fields, potentials, circulation, flux.	2
The theory of Taylor and Laurent series. Constructing series that converge in certain circular or annular regions.	4
Classification of zeros and poles.	2
Definition and computation of residues. Using residues to compute integrals around closed loops in multiply-connected domains.	4
Applications of the residue theorem: infinite sums, generating function computations, Infinite Products – sine function, etc.	2
Conformal mappings, linear fractional transformations, Poisson Integral formula, boundary value problems, fluid flow applications.	4

Short Bibliography

Please note that some of the resources listed below were written by the authors of this report.

Standard texts:

> Remark*: The presence of a text on this list is not meant to imply an endorsement of that text, nor is the absence of a particular text from the list meant to be an anti-endorsement. The texts are chosen to illustrate the sorts of texts that support various types of complex analysis courses.*

J. Bak and D. J. Newman, *Complex Analysis*, 3rd ed., Springer UTM, New York, 2010.

R. V. Churchill and J. W. Brown, *Complex Variables and Applications*, 8th ed.,, McGraw-Hill Book Co., New York, 2009.

S. Fisher, *Complex Variables*, 2nd ed., Dover Publications Inc., New York, 1999.

J. E. Marsden and M. J. Hoffman, *Basic Complex Analysis*, 3rd ed., W. H. Freeman, 1998.

E. B. Saff and D. Snider, *Fundamentals of Complex Analysis with Applications to Engineering, Science, and Mathematics*, 3rd ed., Prentice Hall, New Jersey 2003.

R. A. Silverman, *Complex Analysis with Applications*, Dover Publications Inc., New York, 1984.

D. G. Zill and P. D. Shanahan, *A First Course in Complex Analysis with Applications*, 2nd ed., Jones and Bartlett Publishers, Boston-Toronto-London-Singapore, 2009.

This textbook is great for an introductory course in complex variables with a wide variety of students from math majors to engineering. The book covers standard computations and helps students develop some intuition on how various functions behave. Even though main theorems are proven, students do not need to have a strong background in proof.

Modern & Supplemental Texts

M. A. Brilleslyper, M. J. Dorff, J. M. McDougall, J. S. Rolf, L. E. Schaubroeck, R. L. Stankewitz, and K. Stephenson, *Explorations in Complex Analysis*, Mathematical Association of America, Washington, DC, 2009.

R. Devaney, *An Introduction to Chaotic Dynamical Systems*, 2nd ed., Westview Press, 2003.

This is a foundational book for complex dynamics, covering topics like Julia sets and the Mandelbrot set. It is a great resource for a complex variables class that has a complex dynamical systems focus.

T. Needham, *Visual Complex Analysis*, Oxford University Press, Oxford, UK, 1997.

Open Source Materials

M. Beck, G. Marchesi, D. Pixton, and L. Salbalka, *A First Course in Complex Analysis*, http://math.sfsu.edu/beck/complex.html

G. Cain, *Complex Analysis*, http://people.math.gatech.edu/~cain/winter99/complex.html

http://www.sagemath.org

Websites and Applets

Applets by Jim Rolf that allow for exploration of complex functions, iteration, Julia sets, and the Mandelbrot set can be found here.

This collection of applets makes it possible for students to explore the behavior of a variety of complex functions. The most basic of the applets provided is ComplexTool. In this applet, there is an image of the complex domain on the left and the complex domain on the right. Students are able to explore what happens different regions and curves in the domain when basic complex functions are applied. There are other applets as well that address topics like vectors fields, Julia sets, the Mandelbrot set, and a variety of other topics mentioned in the book, *Explorations in Complex Analysis*, listed above.

Overview of Mandlebrot Sets and Julia Sets on Bob DeVaney's web page.

This provides a very basic background to the Mandelbrot set and Julia sets. It is written at a level that a high school student could understand, but is also of interest to students with much more background. Nice graphics are included.

Mobius Transformations Revealed

This is a YouTube video that visually shows the connection between Mobius transformations in the plane and simple motions of a corresponding sphere. The first part of the video just shows what Mobius transformations look like. The second part brings in the sphere.

Bob Devaney's Mandelbrot set and Julia set Explorer

This applet makes it possible for students to generate Julia sets for the family $f(z) = z^2 + c$ and look at how they correspond to the Mandelbrot set. There is also a feature that allows students to create animations of how the Julia sets change as the value "c" changes in the Mandelbrot set.

Terrance Tao's website contains some useful applets.