Linear Algebra

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Introduction

"I believe that linear algebra is *<u>the</u>* *most important subject in college mathematics. Isaac Newton would not agree! But he isn't teaching mathematics in the 21st century*." Gilbert Strang [5].

Strang is not alone in thinking that linear algebra has a prominent place in today's undergraduate mathematics curriculum. Linear algebra, calculus and data analysis are the only three specific content areas recommended for all mathematical science majors in this *Guide.* From the *Overview*:

> Linear algebra, the study of multivariate linear systems and transformations, is essential preparation for advanced work in the sciences, statistics, and computing. Linear algebra also introduces students to discrete mathematics, algorithmic thinking, a modicum of abstraction, moderate sophistication in notation, and simple proofs. Linear algebra … helps students develop facility with visualization, see connections among mathematical areas, and appreciate the power of abstract thinking.

Technology facilitates solving large systems of linear equations quickly and efficiently. Thus, first courses in linear algebra allow students (as the *Overview* puts it) to "link applications and theory" and also "to use technology effectively, both as a tool for solving problems and as an aid to exploring mathematical ideas." Linear algebra is also especially useful in helping students learn to "use and compare analytical, visual, and numerical perspectives in exploring mathematics."

Prerequisites

The most frequent prerequisite for a Linear Algebra course is Calculus II. But Linear Algebra is taught in so many ways, with such widely varying goals, that other prerequisites, such as Calculus I and Discrete Mathematics, are common. Other institutions require that students complete Multivariable Calculus before Linear Algebra—or *vice versa*. A few institutions require some specific background in a computer language or package designed to treat linear algebra computations and applications. In practice individual institutions must choose prerequisites that best fit their own goals, student populations, and local resources.

Who takes Linear Algebra?

Few subjects other than calculus can rival linear algebra in serving many areas of study and application and in attracting a wide variety of students. Mathematics majors certainly need linear algebra. So, increasingly, do students majoring in mathematics education, physics, chemistry, mathematical biology, computer science, various engineering fields, business, economics, and finance. In recent decades students have brought increasing diversity of background and interest to Linear Algebra classrooms. This sea change has important implications both for content and for pedagogy in Linear Algebra courses.

Computation and the evolving Linear Algebra curriculum

In 1997 Carl C. Cowen received the MAA's Deborah and Franklin Tepper Haimo Award for distinguished College or University Teaching. In his Haimo lecture, titled *The Centrality of Linear Algebra in the Curriculum* [4], Cowen argued that because "no serious application of linear algebra happens without a computer," computation should be part of every beginning Linear Algebra course.

Linear Algebra courses as we now know them became common in the mathematics curriculum only in the late 1960s or early 70s. Cowen notes: "... [E]ngineers have known for more than a century that many problems could be modeled as systems of linear equations or as eigenvalue problems. But what would be the point? Even in the 50s few engineers could hope to solve a system of 100 equations in 100 unknowns; linear algebra was really irrelevant!" As *Matlab* and similar programs came into common usage, faculty in many disciplines began to advise an increasingly diverse population of students to take Linear Algebra. This development caused "many colleges and universities to change from courses dominated by proofs of theorems about abstract vector spaces to courses emphasizing matrix computations and the theory to support them." While the increasing applicability of linear algebra does not require that we stop teaching theory, Cowen argues that "it should encourage us to see the role of the theory in the subject as it is applied."

Beyond supporting the study of substantive applications, computing technology lets students and faculty focus on linear algebra concepts rather than calculations. With matrix manipulation software students can explore open-ended questions that motivate core principles; with computer visualization, students can connect geometric and algebraic aspects of the subject.

Technology can be incorporated in various ways in Linear Algebra courses. In lecture-based courses, technology can appear either interwoven into lectures or in separate lab times. In a lab-based course students might have access to computers during the class period; in courses that use online resources, course technology is readily available. We strongly recommend incorporating technology in one or more forms, while acknowledging that doing so can be labor intensive, especially in a first course iteration.

Core content. Linear Algebra courses are now offered in many "flavors" for many disciplines. Indeed, "Linear Algebra" now connotes more a broad category than a single course, and so it is difficult to make fully general recommendations. Still, the various flavors of Linear Algebra do share important features and explore essentially the same core topics (see further details on topics below).

Pedagogy. Every linear algebra course should develop the core topics in an instructional framework that promotes conceptual understanding of fundamental ideas. Regardless of the course's particular emphasis, assignments should help students build effective thinking skills, as recommended in the *Overview*: critical thinking, pattern recognition, formulating conjectures, writing, student engagement, creation of examples to illustrate a concept, technology, reading proofs, introduction to construction of proofs, and counterexamples. Problem solving arises in both computational (as in matrix computations) and theoretical (as in writing proofs) forms. All linear algebra courses should stress visualization and geometric interpretation of theoretical ideas in 2- and 3-dimensional spaces. Doing so highlights "algebraic and geometric" as "contrasting but complementary points of view," as suggested in the *Overview*. Course projects are especially useful in helping students develop written and oral communication skills.

Learning goals. Students who master the core content of a linear algebra course should attain some specific learning goals. They should be able to

- compute with and recognize properties of particular matrices;
- formulate, solve, apply, and interpret properties of linear systems;
- recognize and use basic properties of subspaces and vector spaces;
- determine a basis and the dimension of a finite-dimensional space;
- find the eigenvalues and eigenvectors of a matrix, and use them to represent a linear transformation;
- recognize and use equivalent forms to identify matrices and solve linear systems;
- read proofs with understanding;
- use definitions and theorems to prove basic results in core topics;
- recognize and use equivalent statements regarding invertible matrices, pivot positions, and solutions of homogeneous systems;
- decide whether a linear transformation is one-to-one or onto and how these questions are related to matrices.

Applications. Every linear algebra course should incorporate interesting applications, both to highlight the broad usefulness of linear algebra and to help students see the role of the theory in the subject as it is applied. Attractive applications may also entice students majoring in other disciplines to choose a minor or additional major in mathematics. While most instructors will choose topics they like or judge appropriate to their students' backgrounds, certain elementary applications and extensions frequently appear in texts and have the reputation of revealing fundamental ideas. Among these, in no particular order, are Markov chains, graph theory, correlation coefficients, cryptology, interpolation, long-term weather prediction, the Fibonacci sequence, difference equations, systems of linear differential equations, network analysis, linear least squares, graph theory, Leslie population models, the power method of approximating the dominant eigenvalue, linear programming, computer graphics, coding theory, spectral decomposition, principal component analysis, discrete and continuous dynamical systems, iterative solutions of linear systems, image processing, and traffic flow.

Core topics. Instructors may vary the given order, or omit topics not appropriate for a given audience.

- Matrices and linear systems: matrix operations and properties; special matrices; linear systems of equations; echelon forms and Gaussian elimination; matrix transformations (with geometric illustrations); matrix inverses; properties of vectors in R^n ; LU-decomposition
- Determinants: computation and properties of determinants, including relation to products, inverses, and transposes; geometric interpretations (Not everyone sees determinants as "core" material in a linear algebra course; see, e.g., [1].)
- Vector spaces: vectors; subspaces; linear independence; basis and dimension; row and column spaces; rank; rank-nullity theorem
- Linear transformations: matrix representations; change of basis; functional properties; kernel and range
- Eigenvalues and eigenvectors: definitions, examples, and properties; computational methods; diagonalization of matrices
- Inner products: norms; orthogonality; orthogonal bases; Gram-Schmidt orthogonalization (optional depending upon time and course goals)

Curricular variations on a theme: **Sample syllabi**

Some institutions offer several linear algebra courses, serving various goals. Here we list some possible examples.

Linear Algebra with Computer Lab

> Goals: Provide a foundation for topics in linear algebra together with applications. Integrate the use of MATLAB software within course topics and applications. Teach students to read proofs effectively, and to construct elementary proofs involving matrices and linear combinations.
>
> Labs: Once a week, two hours; using MATLAB. Computational lab activities promote understanding both of basic concepts from algebraic, symbolic, and geometric viewpoints, and of linear algebra applications.

Differential Equations with Linear Algebra

> Goals: This fast-paced course, primarily in ordinary differential equations but emphasizing the use of linear algebra, has two main objectives: (1) to teach students how to solve linear differential equations and systems thereof; (2) to introduce students to linear algebra concepts, including the majority of core topics. Students also learn to read and construct elementary proofs involving matrices and linear combinations.

Theoretical Linear Algebra: A Second Course

> Goals: This second course adopts a higher degree of abstraction than would a traditional

first course. Students aim to achieve full understanding of the material, to prove key theorems, and to solve challenging problems.

Topics: Major topics, covered more deeply than in a first course, include vector spaces, linear transformations, determinants, eigenvalues and eigenvectors, canonical forms, inner product spaces, the finite-dimensional spectral theorem, singular value decomposition, and bilinear forms. Other topics can be chosen by the instructor.

Several linear algebra course varieties can be directed at future engineers. Computer use is essential.

Linear Algebra for Engineers

Topics include matrices, determinants, vector spaces, eigenvalues and eigenvectors, orthogonality and inner product spaces; applications include brief introductions to difference equations, Markov chains, and systems of linear ordinary differential equations.

Linear Algebra and Vector Analysis for Engineers

Topics include matrix theory, linear equations, Gaussian elimination, determinants, eigenvalue problems and first order systems of ordinary differential equations, vector field theory, theorems of Green, Stokes, and Gauss.

Linear Algebra with Application to Engineering Computations

Topics include basis, linear independence, column space, null space, rank, norms and condition numbers, projections, and matrix properties**.** Students learn to solve matrix-vector systems and consider direct and iterative solvers for non-singular linear systems of equations—their accuracy, convergence properties, and computational efficiency. The course covers under- and over-determined linear systems, and nonlinear systems of equations, as well as eigenvalues, eigenvectors, and singular values--their application to engineering problems.

Linear Algebra and Differential Equations for Engineers

Content areas include matrix theory, eigenvectors and eigenvalues, ordinary and partial differential equations. This course has three parts: matrix algebra (core type topics); systems of linear differential equations; PDEs and Fourier series.

References

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Curriculum Study Group Recommendations for the First Course in Linear Algebra," College Mathematics Journal vol. 24 (1993): 41-46. (Note: this is one article in an issue devoted entirely to the teaching of linear algebra)

3. Carlson, D., Johnson, C. R., Lay, D. C., Porter, A. D., Watkins, A., and Watkins, W., eds. 1997. Resources for Teaching Linear Algebra. Washington, DC: MAA.

4. Cowen, Carl C., "On the Centrality of Linear Algebra in the Curriculum", Remarks on receiving the Deborah and Franklin Tepper Haimo Award for Distinguished College or University Teaching of Mathematics, San Diego California, January 1997.

5. Gilbert Strang, "Linear algebra: A happy chance to apply mathematics", *Proc. Int. Congress on Math. Education* (ICME-10), Denmark (2004).