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William Lowell **PUTNAM** Mathematical Competition

Problems for Session A

The 84th William Lowell Putnam Mathematical Competition 2023

A1 For a positive integer n, let $f_n(x) = \cos(x)\cos(2x)\cos(3x)\cdots\cos(nx)$. Find the smallest n such that $|f_n''(0)| > 2023$.

A2 Let n be an even positive integer. Let p be a monic, real polynomial of degree $2n$; that is to say, $p(x) = x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1 x + a_0$ for some real coefficients $a_0, \dots, a_{2n-1}$. Suppose that $p(1/k) = k^2$ for all integers k such that $1 \leq |k| \leq n$. Find all other real numbers x for which $p(1/x) = x^2$.

A3 Determine the smallest positive real number r such that there exist differentiable functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfying

(a) $f(0) > 0$,

(b) $g(0) = 0$,

(c) $|f'(x)| \leq |g(x)|$ for all x,

(d) $|g'(x)| \leq |f(x)|$ for all x, and

(e) $f(r) = 0$.

A4 Let $v_1, \dots, v_{12}$ be unit vectors in $\mathbb{R}^3$ from the origin to the vertices of a regular icosahedron. Show that for every vector $v \in \mathbb{R}^3$ and every $\varepsilon > 0$, there exist integers $a_1, \dots, a_{12}$ such that $\|a_1 v_1 + \cdots + a_{12} v_{12} - v\| < \varepsilon$.

A5 For a nonnegative integer k, let $f(k)$ be the number of ones in the base 3 representation of k. Find all complex numbers z such that

$$\sum_{k=0}^{3^{1010}-1} (-2)^{f(k)} (z+k)^{2023} = 0.$$

A6 Alice and Bob play a game in which they take turns choosing integers from 1 to n. Before any integers are chosen, Bob selects a goal of "odd" or "even". On the first turn, Alice chooses one of the n integers. On the second turn, Bob chooses one of the remaining integers. They continue alternately choosing one of the integers that has not yet been chosen, until the nth turn, which is forced and ends the game. Bob wins if the parity of $\{k :$ the number k was chosen on the kth turn$\}$ matches his goal. For which values of n does Bob have a winning strategy?