# 2023 Session A

**A1.** For a positive integer $n$, let $f_n(x) = \cos(x)\cos(2x)\cos(3x)\cdots\cos(nx)$. Find the smallest $n$ such that $|f_n''(0)| > 2023$.

**Answer:** 18

**Solution 1:** The Taylor series is

$$\begin{aligned} f_n(x) &= \left(1 - \frac{x^2}{2} + \cdots\right)\left(1 - \frac{(2x^2)}{2} + \cdots\right)\cdots\left(1 - \frac{(nx)^2}{2} + \cdots\right) \\ &= 1 - \frac{x^2}{2}\left(1^2 + 2^2 + \cdots + n^2\right) + \cdots . \end{aligned}$$

Therefore (using the well-known summation formula for sums of squares)

$$f_n''(0) = -(1^2 + 2^2 + \cdots + n^2) = -\frac{n(n+1)(2n+1)}{6}.$$

The question is then to find the minimum $n$ such that $\frac{n(n+1)(2n+1)}{6} > 2023$. One can calculate that this occurs at $n = 18$, where the sum is $3 \cdot 19 \cdot 37 = 2109$ (and at $n = 17$ it is 1785).

**Solution 2:** By the product rule,

$$\begin{aligned} f_n'(x) &= -\sin(x)\cos(2x)\cos(3x)\cdots\cos(nx) - 2\cos(x)\sin(2x)\cos(3x)\cdots\cos(nx) \\ &\quad - \cdots - n\cos(x)\cos(2x)\cdots\cos((n-1)x)\sin(nx) \\ &= -f_n(x)\left(\tan(x) + 2\tan(2x) + \cdots + n\tan(nx)\right) \end{aligned}$$

for $x$ sufficiently small that all the tangents are well-defined. Applying the product rule again and substituting $x = 0$,

$$\begin{aligned} f_n''(0) &= -f_n'(0)\left(\tan(0) + 2\tan(2\cdot 0) + \cdots + n\tan(n\cdot 0)\right) \\ &\quad - f_n(0)\left(\sec^2(0) + 4\sec^2(2\cdot 0) + \cdots + n^2\sec^2(n\cdot 0)\right) \\ &= -(1 + 4 + \cdots + n^2). \end{aligned}$$

One can compute directly (or using the formula in Solution 1) that $|f_{17}''(0)| = 1 + 4 + \cdots + 17^2 = 1785$ and $|f_{18}''(0)| = 1 + 4 + \cdots + 18^2 = 1785 + 324 = 2109$, so the answer is 18.

**A2.** Let $n$ be an even positive integer. Let $p$ be a monic, real polynomial of degree $2n$; that is to say, $p(x) = x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1x + a_0$ for some real coefficients $a_0, \ldots, a_{2n-1}$. Suppose that $p(1/k) = k^2$ for all integers $k$ such that $1 \le |k| \le n$. Find all other real numbers $x$ for which $p(1/x) = x^2$.

**Answer:** $\pm 1/n!$

**Solution 1:** The given condition can be equivalently written as $p(x) = \frac{1}{x^2}$ for $x = \pm\frac{1}{k}, k = 1, \ldots, n$. Now define $g(x) := x^2p(x) - 1$, and note that $p(1/x) = x^2$ is equivalent to $g(1/x) = 0$. Notice that $g$ is a monic polynomial of degree $2n+2$, and by the preceding observation, it has roots at all $x = \pm\frac{1}{k}$. Unique factorization of polynomials (and/or the Fundamental Theorem of Algebra) then implies that

$$\begin{aligned} g(x) &= (x-1)(x+1)\left(x - \frac{1}{2}\right)\left(x + \frac{1}{2}\right)\cdots\left(x - \frac{1}{n}\right)\left(x + \frac{1}{n}\right)\cdot(x^2 + ax + b) \\ &= \left(x^2 - 1\right)\left(x^2 - \frac{1}{4}\right)\cdots\left(x^2 - \frac{1}{n^2}\right)\cdot(x^2 + ax + b), \end{aligned}$$

where $a$ and $b$ are real numbers.

In order to determine these final coefficients, first note that by definition of $g(x)$, the coefficient of $x$ is zero. But on the other hand, this coefficient is $\dfrac{(-1)^n}{n!^2}a$, so $a = 0$. Now consider the value at $x = 0$, which gives

$$g(0) = -1 = \frac{(-1)^n}{n!^2}b.$$

We therefore conclude (using that $n$ is even) that $b = -n!^2$. In all,

$$g(x) = \left(x^2 - 1\right)\left(x^2 - \frac{1}{4}\right)\cdots\left(x^2 - \frac{1}{n^2}\right)\cdot\left(x^2 - n!^2\right).$$

Finally, we see that $g(1/x)$ has the additional roots $x = \pm\frac{1}{n!}$.

**Solution 2:** We first show that $p$ is even.

**Claim.** *Suppose that $q$ is a monic, degree $2n$ polynomial. If there exists a sequence of distinct positive values $x_1, \ldots, x_n$ such that $q(x_j) = q(-x_j)$ for $1 \le j \le n$, then $q$ is even.*

*Proof.* The polynomial $q(x) - q(-x)$ has degree at most $2n - 1$, but has (at least) $2n$ roots $\pm x_1, \ldots, \pm x_n$. Therefore, $q(x) - q(-x)$ is identically zero, so $q$ is even. $\square$

Thus $p(x) = s(x^2)$, where $s$ is a monic, degree $n$ polynomial such that $s(1/k^2) = k^2$ for $1 \le k \le n$. Let $h(x) := x \cdot s(x) - 1$. Then $h$ is a monic, degree $n + 1$ polynomial with roots at $\frac{1}{k^2}$, so

$$h(x) = (x-1)\left(x - \frac{1}{4}\right)\cdots\left(x - \frac{1}{n^2}\right)(x + b)$$

for some real number $b$. Plugging in $x = 0$ gives

$$h(0) = -1 = \frac{(-1)^n}{n!^2}b.$$

Finally, substituting $x^2$ for $x$ gives $h(x^2) = x^2p(x) - 1$, and the remainder of the proof proceeds as in Solution 1.

*Remark.* Although this is not needed in the proof of the claim, one can use Lagrange interpolation to say slightly more. For $1 \leq a_j \leq n$, let $a_j := q(\pm x_j)$. Define $L(x)$ as the unique polynomial of degree strictly less than $n$ with the values $L(x_j^2) = a_j$ for all $j$ (Lagrange interpolation gives a formula for $L$). Then

$$q(x) = \left(x^2 - x_1^2\right) \cdots \left(x^2 - x_n^2\right) + L(x^2).$$

**A3.** Determine the smallest positive real number $r$ such that there exist differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ satisfying

(a) $f(0) > 0$,

(b) $g(0) = 0$,

(c) $|f'(x)| \leq |g(x)|$ for all $x$,

(d) $|g'(x)| \leq |f(x)|$ for all $x$, and

(e) $f(r) = 0$.

**Answer:** $\pi/2$

**Solution 1:** Let $h(x) := f(x)^2 + g(x)^2$. Then, using the AM-GM inequality,

$$\begin{aligned} |h'(x)| &= \left|2(f(x)f'(x) + g(x)g'(x))\right| \leq 2\left(|f(x)||f'(x)| + |g(x)||g'(x)|\right) \\ &\leq 4|f(x)||g(x)| \leq 2\left(f(x)^2 + g(x)^2\right) = 2h(x). \end{aligned}$$

Thus, $h(x)e^{2x}$ has a nonnegative derivative. For $x \geq 0$, it follows that $h(x)e^{2x} \geq h(0)e^0 = f(0)^2$, so $h(x) \geq f(0)^2 e^{-2x} > 0$.

Now define $\theta(x) := \tan^{-1} \frac{g(x)}{f(x)}$ on the largest interval around $x = 0$ on which $f(x) > 0$ (this includes a neighborhood of 0 by continuity of $f$). Then

$$\theta'(x) = \frac{1}{1 + \frac{g(x)^2}{f(x)^2}} \cdot \frac{g'(x)f(x) - g(x)f'(x)}{f(x)^2} = \frac{g'(x)f(x) - g(x)f'(x)}{f(x)^2 + g(x)^2},$$

which implies that $|\theta'(x)| \leq 1$, and (since $\theta(0) = 0$) therefore $|\theta(x)| \leq x$ for $x \geq 0$ such that $\theta$ is defined.

Finally, observe that $f(x)^2 = h(x)\cos^2\theta(x)$, and therefore $f(x)^2 \geq f(0)^2 e^{-2x}\cos^2 x > 0$ for $x \in (0, \pi/2)$. Thus, $r \geq \pi/2$. Letting $f(x) = \cos x$ and $g(x) = \sin x$ shows that $r = \pi/2$ is possible.

**Solution 2:** Notice that $f(x) = \cos x$ and $g(x) = \sin x$ satisfy all the conditions of the problem with $r = \pi/2$.

To see that no smaller value of $r$ is possible, we claim that $f(x) \geq f(0)\cos x$ for $0 \leq x < \pi/2$. If not, then there is some $z \in (0, \pi/2)$ such that $f(z)/\cos z < f(0)$. Since $f(x)/\cos x$ is continuous for $0 \leq x < \pi/2$ and $f(0)/\cos 0 = f(0) > 0$, we can choose $z$ such that $f(x)/\cos x > 0$ [whence $f(x) > 0$] for $0 \leq x \leq z$. Since $f(x)/\cos x$ is differentiable for $0 < x < z$, the Mean Value Theorem implies that its derivative must be negative at some $y \in (0, z)$. Thus, $f'(y)\cos y + f(y)\sin y < 0$. Since $|f'(y)| \leq |g(y)|$ and $\cos y > 0$, it follows that $f(y)\sin y - |g(y)|\cos y < 0$.

Let $h(x) = f(x)\sin x - |g(x)|\cos x$. Since $h(0) = 0$, $h(y) < 0$, and $h$ is continuous, there is a greatest value $w \in [0, y)$ such that $h(w) \geq 0$. Then for $w < x < y$, we have $h(x) < 0$ and $f(x)\sin x \geq 0$ [since $f(x) > 0$ for $0 \leq x \leq z$ and $y < z$], so $|g(x)|\cos x > 0$, and thus $|g(x)| > 0$, for all such $x$. In particular, $h$ is differentiable on the interval $(w, y)$, so by the Mean Value Theorem $h'(x) < 0$ for some $x \in (w, y)$. Further, since $g$ is continuous and nonzero on $(w, y)$, it does not change sign on this interval; without loss of generality, assume that $g$ is positive on $(w, y)$ [otherwise, replace $g$ with $-g$]. Then $h'(x) = (f'(x) + g(x)) \sin x + (f(x) - g'(x)) \cos x$. Since $h'(x) < 0$, $\sin x > 0$, and $\cos x > 0$, we must have $f'(x) + g(x) < 0$ or $f(x) - g'(x) < 0$. But since $f(x) > 0$ and $g(x) > 0$, this would require either $|f'(x)| > g(x) = |g(x)|$ or $|g'(x)| > f(x) = |f(x)|$, a contradiction.

**A4.** Let $v_1, \ldots, v_{12}$ be unit vectors in $\mathbb{R}^3$ from the origin to the vertices of a regular icosahedron. Show that for every vector $v \in \mathbb{R}^3$ and every $\varepsilon > 0$, there exist integers $a_1, \ldots, a_{12}$ such that $\|a_1 v_1 + \cdots + a_{12} v_{12} - v\| < \varepsilon$.

**Solution 1:** We first claim that the vertices of a regular pentagon centered at the origin in $\mathbb{R}^2$ generate a dense additive subgroup. Identify $\mathbb{R}^2 \cong \mathbb{C}$, and assume without loss of generality that the vertices are the fifth roots of unity $\zeta_5^n = e^{\frac{2\pi i n}{5}}$. Define

$$r := \zeta_5 + \zeta_5^4 = 2\cos\frac{2\pi}{5} < 2\cos\frac{\pi}{3} = 1$$

(in fact, $r = \frac{\sqrt{5}-1}{2}$, which follows from observing that $r^2 + r = 1$). Then the positive powers of $r$ accumulate at 0, and are all contained in the subring $\mathbb{Z}[\zeta_5]$. Therefore this subring is dense in $\mathbb{R}$, since it contains all $\{mr^n \mid m \in \mathbb{Z}, n \geq 0\}$. Furthermore, it similarly contains a dense subset of $\zeta_5\mathbb{R}$, and thus a dense subset of $\mathbb{R} + \zeta_5\mathbb{R} = \mathbb{C}$.

Now suppose that $v_1, \ldots, v_5$ are the neighbors of some fixed vertex $v$ in the icosahedron. Then $v_1 - v_2, v_2 - v_3, \ldots, v_5 - v_1$ are the vertices of a regular pentagon in the plane perpendicular to the line through 0 and $v$. Therefore the set of vertices generates a dense set in this plane, and similarly, in the plane perpendicular to (say) the line through 0 and $v_1$. These two planes span $\mathbb{R}^3$.

**Solution 2:** Write the vertex-vectors as $\pm v_1, \ldots, \pm v_6$ where $v_2, \ldots, v_6$ are each adjacent to $v_1$, and are adjacent to each other in the pairs $(v_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_6), (v_6, v_2)$. Then $v_2$ is also adjacent to $-v_4$ and $-v_5$, etc. Since all sides of the icosahedron have the same length and $v_j \cdot v_j = 1$ for all $j$, the value of $v_j \cdot v_k$ is the same for all pairs of adjacent vertices $(v_j, v_k)$. Thus,

$$\begin{aligned} v_2 \cdot v_3 &= v_3 \cdot v_4 = v_4 \cdots v_5 = v_5 \cdot v_6 = v_6 \cdot v_2 \\ &= -v_2 \cdot v_4 = -v_2 \cdot v_5 = -v_3 \cdot v_5 = -v_3 \cdot v_6 = -v_4 \cdot v_6. \end{aligned}$$

Then the cross terms $2v_j \cdot v_k$ cancel each other in the following calculation:

$$(v_2 + \cdots + v_6) \cdot (v_2 + \cdots + v_6) = v_2 \cdot v_2 + \cdots + v_6 \cdot v_6 = 5.$$

By symmetry and adjacency, $v_2 + \cdots v_6$ is a positive multiple of $v_1$, so $v_2 + \cdots v_6 = \sqrt{5} v_1$. By Kronecker's Theorem on Diophantine approximation, the integer linear combinations of 1 and $\sqrt{5}$ are dense in the reals, so the integer linear combinations of $v_1$ and $v_2 + \cdots + v_6$ are dense in the line spanned by $v_1$. By the analogous argument, appropriate integer linear combinations of $v_1, \ldots, v_6$ are dense in the line spanned by $v_2$ and in the line spanned by $v_3$. Since $v_1, v_2, v_3$ are not coplanar, they span three-space. Thus, every vector $v$ in three-space can be written $c_1 v_1 + c_2 v_2 + c_3 v_3$, and since each term in this sum can be approximated arbitrarily closely by an integer linear combination of $v_1, \ldots, v_6$, so can $v$.

**A5.** For a nonnegative integer $k$, let $f(k)$ be the number of ones in the base 3 representation of $k$. Find all complex numbers $z$ such that

$$\sum_{k=0}^{3^{1010}-1} (-2)^{f(k)}(z+k)^{2023} = 0.$$

**Answer:** $\dfrac{1-3^{1010}}{2}$, $\dfrac{1-3^{1010}}{2} \pm \dfrac{\sqrt{3^{2020}-1}}{4}i$

**Solution 1:** If $n$ is an integer, a *quasi-base-3* representation of $n$ is $n = a_N 3^N + a_{N-1}3^{N-1} + \cdots + a_1 \cdot 3 + a_0$, where all $a_j \in \{-1, 0, 1\}$. This can also be written in the shorthand $(a_N a_{N-1} \cdots a_0)_3$, with parentheses used as needed for clarity. For example, $7 = 1(-1)1_3$. If the leading digit is required to be $\pm 1$, then it is a standard fact that this representation is unique (both existence and uniqueness are easily proven by induction). However, here we will also be interested in representations with some number of leading zeros. Let $A_N := \dfrac{3^N-1}{2} = \underbrace{11\cdots 1}_{N \text{ digits}}{}_3$. It can similarly be shown that if $|n| \leq A_N$, then $n$ has a unique quasi-base-3 representation consisting of exactly $N$ digits.

Define $f_{0,N}(\ell)$ to be the number of zeros when $\ell$ is written in its $N$-digit quasi-base-3 representation. For example, $f_{0,3}(8) = 1$ and $f_{0,5}(8) = 3$, as the left-extended quasi-base-3 expansion is $8 = \ldots 0010(-1)_3$.

**Claim.** *Let $u = z + A_N$. The sum in the problem statement is equivalent to (with $N = 1010$)*

$$S_N(u) := \sum_{\ell=-A_N}^{A_N} (-2)^{f_{0,N}(\ell)}(u+\ell)^{2N+3}. \tag{1}$$

This follows from the summation shift $k = \ell + A_N$, as well as the straightforward observation that $f(\ell + A_N) = f_{0,N}(\ell)$ (note that $\ell \mapsto \ell + A_N$ is a bijection from $N$-digit quasi-base-3 representations to $N$-digit base-3 representations for the range $\ell = -A_N, \ldots, A_N$, while the definition of $f$ is unchanged by the presence of leading zeros).

If $d$ is an integer, define the centered, $d$-shifted second-difference operator by

$$\Delta_d^2(h(x)) := h(x+d) - 2h(x) + h(x-d).$$

This operator satisfies some simple properties that will be helpful later for reducing longer expressions.

**Lemma 1.** *1. $\Delta_d^2$ is a parity-preserving operator on functions (i.e. $\Delta_d^2(h)$ is an even/odd function as $h$ is).*

*2. $\Delta_d^2$ acts on monomials as*

$$\Delta_d^2(x^n) = 2\sum_{1 \leq j \leq \frac{n}{2}} \binom{n}{2j} d^{2j} x^{n-2j}.$$

*3. If $p(x)$ is a polynomial of degree $n$, then $\Delta_d^2(p(x))$ is a polynomial of degree $n-2$ (or is identically zero if $n \leq 1$).*

*Proof.* 1. Suppose that $h$ is an even or odd function. By definition, this means that $h(-x) = \mathrm{sgn}(h) \cdot h(x)$, where $\mathrm{sgn}(h) = \pm 1$ denotes the parity of $h$. Then

$$\begin{aligned}\Delta_d^2(h(-x)) &= h(-x+d) - 2h(-x) + h(-x-d) \\ &= h(-(x+d)) - 2h(-x) + h(-(x-d)) \\ &= \mathrm{sgn}(h) \cdot \Delta_d^2(h(x)).\end{aligned}$$

2. By the Binomial Theorem,

$$\Delta_d^2(x^n) = \sum_{j=1}^{n} \binom{n}{j} d^j x^{n-j} + \sum_{j=1}^{n} \binom{n}{j} (-d)^j x^{n-j} = 2 \sum_{1 \leq j \leq \frac{n}{2}} \binom{n}{2j} d^{2j} x^{n-2j}.$$

3. This follows by considering only the highest degree term in the sum from part 2.

□

We next show that sums of the form (1) can be written in terms of the second-difference operator.

**Lemma 2.** *If $N$ is a positive integer, then*

$$\Delta_{3^{N-1}}^2 \cdots \Delta_3^2 \Delta_1^2 h(x) = \sum_{\ell=-A_N}^{A_N} (-2)^{f_{0,N}(\ell)} h(x+\ell).$$

*Proof.* The base case of $N = 0$ simply states that $h(x) = h(x)$. For the inductive step,

$$\begin{aligned}&\Delta_{3^N}^2 \sum_{\ell=-A_N}^{A_N} (-2)^{f_{0,N}(\ell)} h(x+\ell) \\ &= \sum_{\ell=-A_N}^{A_N} (-2)^{f_{0,N}(\ell)} \left(h\left(x + 3^N + \ell\right) - 2h(x+\ell) + h\left(x - 3^N + \ell\right)\right) \\ &= \sum_{\ell=3^N-A_N}^{3^N+A_N} (-2)^{f_{0,N}(\ell)} h(x+\ell) + \sum_{\ell=-A_N}^{A_N} (-2)^{f_{0,N}(\ell)+1} h(x+\ell) \\ &\qquad + \sum_{\ell=-3^N-A_N}^{-3^N+A_N} (-2)^{f_{0,N}(\ell)} h(x+\ell) \\ &= \sum_{\ell=-A_{N+1}}^{A_{N+1}} (-2)^{f_{0,N+1}(\ell)} h(x+\ell).\end{aligned}$$

The final line follows because in the first and third sum $\ell$ has leading coefficient $\pm 1$, so $f_{0,N+1}(\ell) = f_{0,N}(\ell)$, whereas in the middle sum the leading coefficient is 0, so $f_{0,N+1}(\ell) = f_{0,N}(\ell) + 1$. Furthermore, the three summation ranges combine to one because $A_{N+1} = 3^N + A_N$ by definition. □

In particular, Lemma 2 implies that

$$S_N(u) = \Delta^2_{3^{N-1}} \cdots \Delta^2_3 \Delta^2_1 u^{2N+3}, \tag{2}$$

with the implicit notational assumption that the $\Delta^2$ operators now act on the variable $u$. Lemma 1 parts 1 and 3 then imply that $S_N(u)$ reduces to an odd, cubic polynomial (i.e. of the form $au^3 + bu$). The exact coefficients can now be determined using Lemma 1 part 2.

In particular, only the two highest-order terms are relevant. For $n \geq 4$, the Lemma states that

$$\Delta^2_d(x^n) = 2\binom{n}{2} d^2 x^{n-2} + 2\binom{n}{4} d^4 x^{n-4} + O\left(x^{n-6}\right).$$

For a nonnegative integer $m$ and real number $x$, the *falling factorial* is defined by $(x)_m := x(x-1)(x-2)\cdots(x-m+1)$.

**Lemma 3.** *If $N$, $k$ and $d$ are positive integers, where $k \geq 4$, then*

$$\begin{aligned}
&\Delta^2_{d^{N-1}} \cdots \Delta^2_d \Delta^2_1 x^{2N+k} \\
&\quad = (2N+k)_{2N} \cdot d^{N(N-1)} \left[ x^k + \frac{k(k-1)}{3 \cdot 4} \left(1^2 + d^2 + \cdots + d^{2(N-1)}\right) x^{k-2} + O\left(x^{k-4}\right) \right].
\end{aligned}$$

*The statement is also true for $k \leq 3$, except that any monomials in $x$ with negative exponents do not appear.*

*Proof.* We induct on $N$. The base case is $N = 1$, and by Lemma 1,

$$\Delta^2_1 x^{2+k} = (2+k)_2 \cdot 1^2 x^k + \frac{(2+k)_4}{3 \cdot 4} 1^4 x^{k-2}.$$

For the inductive step, suppose that the statement is true for $N$. Then we have (replacing $k$ by $2+k$ when applying the statement for $N$)

$$\begin{aligned}
&\Delta^2_{d^N} \Delta^2_{d^{N-1}} \cdots \Delta^2_1 x^{2(N+1)+k} \\
&= \Delta^2_{d^N} \Big( (2(N+1)+k)_{2N} \cdot d^{N(N-1)} \Big[ x^{2+k} \\
&\qquad\qquad + \frac{(2+k)(1+k)}{3 \cdot 4} \left(1^2 + d^2 + \cdots + d^{2(N-1)}\right) x^k + O\left(x^{k-2}\right) \Big] \Big) \\
&= (2(N+1)+k)_{2N} \cdot d^{N(N-1)} \Big[ \Big( (2+k)(1+k) d^{2N} x^k + \frac{(2+k)_4}{3 \cdot 4} d^{4N} x^{k-2} \Big) \\
&\qquad + \frac{(2+k)(1+k)}{3 \cdot 4} \left(1^2 + d^2 + \cdots + d^{2(N-1)}\right) \cdot k(k-1) d^{2N} x^{k-2} + O\left(x^{k-4}\right) \Big] \\
&= (2(N+1)+k)_{2N+2} \cdot d^{N(N+1)} \Big[ x^k + \frac{k(k-1)}{3 \cdot 4} \left(1^2 + d^2 + \cdots + d^{2(N-1)} + d^{2N}\right) x^{k-2} \\
&\qquad\qquad\qquad\qquad\qquad\qquad + O\left(x^{k-4}\right) \Big].
\end{aligned}$$

∎

Applying Lemma 3 (with $d = 3$ and $k = 3$) to (2) gives

$$S_N(u) = (2N+3)_{2N} \cdot 3^{N(N-1)} \left( u^3 + \frac{3 \cdot 2}{3 \cdot 4} \frac{3^{2N} - 1}{3^2 - 1} u \right).$$

The cubic $u^3 + \frac{3^{2N}-1}{16}u$ has the roots $u \in \left\{0, \pm\frac{\sqrt{3^{2N}-1}}{4}i\right\}$. Finally, these are translated to roots of the original expression in $z$ using (1).

**Solution 2:** For nonnegative integers $n$, let

$$g_n(z) = \frac{1}{(2n+5)!}\sum_{k=0}^{3^n-1}(-2)^{f(k)}(z+k)^{2n+5},$$

and notice that the equation to be solved is $(2023!)g''_{1010}(z) = 0$. Let $c_n = (3^n - 1)/2$. We will prove by induction on $n$ that

$$g''_n(z) = 3^{n^2-n}\left(\frac{(z+c_n)^3}{6} + \frac{(3^{2n}-1)(z+c_n)}{96}\right).$$

Since $g_0(z) = z^5/5!$, we have $g''_0(z) = z^3/3! = (z+c_0)^3/6$, verifying the base case. Assume now the formula for $g''_n$ above holds for a particular $n \geq 0$. Observe that $f(j+3^n) = f(j)+1$ and $f(j+2\cdot 3^n) = f(j)$ for $0 \leq j < 3^n$. Then, with the substitutions $\ell = k - 3^n$ and $m = k - 2\cdot 3^n$ in the corresponding sums below,

$$\begin{aligned}
g''_{n+1}(z) &= \frac{(2n+7)(2n+6)}{(2n+7)!}\sum_{k=0}^{3^{n+1}-1}(-2)^{f(k)}(z+k)^{2n+5} \\
&= \frac{1}{(2n+5)!}\sum_{k=0}^{3^n-1}(-2)^{f(k)}(z+k)^{2n+5} + \frac{1}{(2n+5)!}\sum_{\ell=0}^{3^n-1}(-2)^{f(\ell)+1}(z+\ell+3^n)^{2n+5} \\
&\quad + \frac{1}{(2n+5)!}\sum_{m=0}^{3^n-1}(-2)^{f(m)}(z+m+2\cdot 3^n)^{2n+5} \\
&= g_n(z) - 2g_n(z+3^n) + g_n(z+2\cdot 3^n) \\
&= \int_0^{3^n} g'_n(z+3^n+t)dt - \int_0^{3^n} g'_n(z+t)dt = \int_0^{3^n}\int_t^{3^n+t} g''_n(z+s)ds\,dt \\
&= \int_0^{2\cdot 3^n}\int_{\max(0,s-3^n)}^{\min(3^n,s)} g''_n(z+s)dt\,ds \\
&= \int_0^{2\cdot 3^n}(3^n - |s-3^n|)g''_n(z+s)ds = \int_{-3^n}^{3^n}(3^n-|u|)g''_n(z+u+3^n)du \\
&= 3^{n^2-n}\int_{-3^n}^{3^n}(3^n-|u|)\left(\frac{(z+u+3^n+c_n)^3}{6} + \frac{(3^{2n}-1)(z+u+3^n+c_n)}{96}\right)dt.
\end{aligned}$$

The integrand is $3^n - |u|$ times a cubic polynomial of $u$. Since $3^n - |u|$ is an even function of $u$, its integral from $-3^n$ to $3^n$ times an odd power of $u$ is zero, so we can eliminate the cubic and linear terms from the cubic polynomial of $u$. Having done so, the integrand becomes an even function of $u$, which we can integrate instead from 0 to $3^n$ and double the result. Using also the fact that $3^n + c_n = c_{n+1}$, we have

$$
\begin{aligned}
g''_{n+1}(z) &= 2 \cdot 3^{n^2-n} \int_0^{3^n} (3^n - u) \left( 3u^2 \frac{(z + c_{n+1})}{6} + \frac{(z + c_{n+1})^3}{6} + \frac{(3^{2n} - 1)(z + c_{n+1})}{96} \right) du \\
&= 2 \cdot 3^{n^2-n} \left( \frac{3^{4n}}{4} \frac{(z + c_{n+1})}{6} + \frac{3^{2n}}{2} \left( \frac{(z + c_{n+1})^3}{6} + \frac{(3^{2n} - 1)(z + c_{n+1})}{96} \right) \right) \\
&= 3^{2n} \cdot 3^{n^2-n} \left( \frac{(z + c_{n+1})^3}{6} + \left( \frac{3^{2n}}{12} + \frac{(3^{2n} - 1)}{96} \right) (z + c_{n+1}) \right) \\
&= 3^{(n+1)^2-(n+1)} \left( \frac{(z + c_{n+1})^3}{6} + \frac{3^{2n+2} - 1}{96} (z + c_{n+1}) \right),
\end{aligned}
$$

completing the induction.

Recall that the answers to the problem are the roots of $g''_{1010}$, which are those $z$ for which $z + c_{1010} = 0$ or $(z + c_{1010})^2 = -(3^{2020} - 1)/16$, yielding the answers given above.

**Solution 3:** For a sequence of integers $b$, set $p(b) = \#\{i : b_i = 0\}$. For nonnegative integers $n, m$, define

$$
h_{n,m}(y) = \sum_{b \in \{-1,0,1\}^n} (-2)^{p(b)} \left( y + \sum_{i=0}^{n-1} b_i 3^i \right)^m
$$

As described in Solution 1, the polynomial in the problem is $h_{n,m}(y)$ for $n = 1010, m = 2n+3$ and $y = z + 1 + 3 + \cdots + 3^{n-1} = z + \frac{3^n - 1}{2}$.

We have that $p(b) = n - \sum_i |b_i|$, and we can expand the polynomial by the Binomial and Multinomial Formulas as follows:

$$
\begin{aligned}
h_{n,m}(y) &= \sum_{b \in \{-1,0,1\}^n} (-2)^{n - \sum |b_i|} \sum_{k=0}^{m} \binom{m}{k} y^{m-k} \left( \sum_{i=0}^{n-1} b_i 3^i \right)^k \\
&= \sum_{k=0}^{m} \binom{m}{k} y^{m-k} (-2)^n \sum_{b \in \{-1,0,1\}^n} (-2)^{-\sum_i |b_i|} \sum_{\substack{a_0 + \cdots + a_{n-1} = k \\ a_j \geq 0}} \binom{k}{a_0, a_1, \ldots} \prod_{i=0}^{n-1} (b_i 3^i)^{a_i} \\
&= \sum_{k=0}^{m} \binom{m}{k} y^{m-k} (-2)^n \sum_{\substack{a_0 + \cdots + a_{n-1} = k \\ a_j \geq 0}} \binom{k}{a_0, a_1, \ldots} \sum_{b_0 \in \{-1,0,1\}} \cdots \sum_{b_{n-1} \in \{-1,0,1\}} \prod_{i=0}^{n-1} (-2)^{-|b_i|} (b_i 3^i)^{a_i} \\
&= \sum_{k=0}^{m} \binom{m}{k} y^{m-k} (-2)^n \sum_{\substack{a_0 + \cdots + a_{n-1} = k \\ a_j \geq 0}} \binom{k}{a_0, a_1, \ldots} \prod_{i=0}^{n-1} \left( \sum_{b_i \in \{-1,0,1\}} (-2)^{-|b_i|} (b_i 3^i)^{a_i} \right)
\end{aligned}
$$

Notice that the terms in the final parentheses evaluate as (recall that the correct convention for powers of 0 in multinomial expansions is $0^0 = 1$, and $0^\ell = 0$ for $\ell \geq 1$)

$$
\sum_{b_i \in \{-1,0,1\}} (-2)^{-|b_i|} (b_i 3^i)^{a_i} = (-2)^{-1} (-3^i)^{a_i} + 0^{a_i} + (-2)^{-1} (3^i)^{a_i} = \begin{cases} 0 & \text{, if } a_i = 0 \text{ or odd} \\ -3^{a_i i} & \text{, if } a_i \geq 2 \text{ is even} \end{cases}
$$

Thus the only nonzero summands can occur only when all $a_i \geq 2$ and are even and so $k = \sum a_i \geq 2n$ and is even. Since $k \leq m = 2n+3$, we have either $k = 2n$ and then $a_i = 2$ for all $i$, or $k = 2m+2$ and then $a_j = 4$ for some $j$ and $a_i = 2$ for all $i \neq j$. The summation simplifies as

$$h_{n,2n+3}(y) = (-2)^n \left( y^3 \binom{2n+3}{2n} \binom{2n}{2,2,\ldots} \prod_{i=0}^{n-1}(-3^{2i}) + y \binom{2n+3}{2n+2} \binom{2n+2}{4,2,2,\cdots} \sum_{j=0}^{n-1}(-3^{4j}) \prod_{\substack{0 \leq i \leq n-1 \\ i \neq j}} (-3^{2i}) \right)$$

$$= (-2)^n(-1)^n \prod_{i=0}^{n-1} 3^{2i} \left( \binom{2n+3}{3} \frac{(2n)!}{2^n} y^3 + y(2n+3) \frac{(2n+2)!}{12 \cdot 2^n} \sum_{j=0}^{n-1} 3^{2j} \right)$$

Factoring out $\frac{(2n+3)!}{6 \cdot 2^n}$ and simplifying the geometric sum over $j$ we are left with looking for the solutions of

$$y^3 + \frac{3^{2n}-1}{16} y = 0,$$

whose roots are $\{0, \pm i \frac{\sqrt{3^{2n}-1}}{4}\}$. Finally, recalling that $z = y - \frac{3^n-1}{2}$, the roots of the original polynomial are then $\{-\frac{3^n-1}{2}, -\frac{3^n-1}{2} \pm i \frac{\sqrt{3^{2n}-1}}{4}\}$ with $n = 2020$.

**A6.** Alice and Bob play a game in which they take turns choosing integers from 1 to $n$. Before any integers are chosen, Bob selects a goal of "odd" or "even". On the first turn, Alice chooses one of the $n$ integers. On the second turn, Bob chooses one of the remaining integers. They continue alternately choosing one of the integers that has not yet been chosen, until the $n$th turn, which is forced and ends the game. Bob wins if the parity of $\{k\colon \text{the number } k \text{ was chosen on the } k\text{th turn}\}$ matches his goal. For which values of $n$ does Bob have a winning strategy?

**Answer:** Bob can always win by choosing the goal that matches the parity of $n$.

**Solution 1:** We say that $k$ is a "fixed point" if $k$ is chosen on the $k$th turn.

If $n$ is even, then Bob can win by following a simple "mirror" strategy. Divide the available numbers into adjacent pairs $(1,2),(3,4),\ldots,(n-1,n)$. Whenever Alice chooses a number from some pair, Bob chooses the other number from the pair on his turn. If on turn $2j+1$ Alice chooses $2j+1$, then Bob also creates a fixed point on turn $2j+2$, thereby adding two to the total number of fixed points. Otherwise, Alice and Bob add zero fixed points on turns $2j+1$ and $2j+2$. There are therefore an even number of fixed points after each of Bob's turns, and since the game ends after the $n$th turn, which is Bob's, he wins the game.

Now suppose that $n = 2m+1$ is odd. For the remainder of the proof, denote Alice's choices by $A_1, A_3, \ldots, A_{2m+1}$, and Bob's by $B_2, \ldots, B_{2m}$. In particular, on the first turn of the game, Alice chooses $A_1$, followed by Bob choosing $B_2$, and so on, and all of the $A$s and $B$s must be distinct integers from 1 to $2m+1$. Let $F_k$ be the number of fixed points after $k$ turns, reduced modulo 2. We have $F_0 = 0$, and Bob wins if $F_n = 1$.

If $n = 1$, then Bob clearly wins. Otherwise, for $m \geq 1$ we claim that Bob wins by playing according to the following rules. Throughout $2j$ will denote Bob's current turn in the game, beginning by applying Rule R1 on turn 2, followed by whichever of Rule R2 or R3 applies on turn $2j$ for $j \geq 2$. The rule statements include several assumed properties that will be justified inductively later, most importantly that $F_{2j-2} = 1$ for $j \geq 2$.

(R1) (a) If $A_1$ is in $\{1,2\}$, then Bob chooses the other integer in this pair as $B_2$. This results in either 0 or 2 fixed points, so $F_2 = 0$. Now rename the remaining integers $3,\ldots,n$ to $1,\ldots,n-2$, and inductively restart the game.

(b) If $A_1 = a \geq 3$, then Bob chooses $B_2 = 2$, so that $F_2 = 1$. If $n = 3$, then $a = 3$ and the game ends with the forced value $A_3 = 1$, so $F_3 = 1$. Otherwise, proceed to the following rules.

(R2) If $j \geq 2$ and $A_{2j-1} = 2j-1$, then (we will show later that $2j \leq a$ in this case):

(a) If $2j < a$, Bob chooses $B_{2j} = 2j$. Then $F_{2j} = F_{2j-2} + 2 \mod 2 = 1$.

(b) If $2j = a$, then Bob chooses $B_{2j} = 1$. Then $F_{2j} = F_{2j-2} + 1 \mod 2 = 0$. Rename the remaining integers $a+1,\ldots,n$ to $1,\ldots,n-a$ and inductively restart the game.

(R3) If $j \geq 2$ and $A_{2j-1} \neq 2j-1$, then Bob chooses $B_{2j} = 2j+1$ if it has not been previously chosen; otherwise, Bob chooses $B_{2j}$ to be an arbitrary unchosen integer not equal to $2j$. Then $F_{2j} = F_{2j-2} = 1$.

Since Rules R1b, R2a, and R3 all end with $F_{2j} = 1$, and the other rules result in a restart that resets $j$ to 1, the assumption that $F_{2j-2} = 1$ for $j \geq 2$ is true by induction. Bob's last turn occurs when $j = m$. If this turn results in a restart (from Rule R1a or R2b), then $F_{2m} = 0$, and Alice is forced to choose a fixed point on her final turn, so $F_n = 1$ and Bob wins. Otherwise, Rule R1b or R2a or R3 applies on Bob's last turn, and in each case $F_{2m} = 1$. Rule R1b explains why $F_n = 1$ in that case. If Rule R2a applies on turn $2m$, then $a = n = 2m + 1$, so Alice cannot choose $A_n = n$, and $F_n = 1$. If Rule R3 applies on turn $2m$, then either Bob chooses $B_{2m} = 2m + 1$, or $2m + 1$ has already been chosen; again, Alice cannot choose $A_n = n$, and $F_n = 1$.

To complete the proof, we need to verify that the strategy above respects the rules of the game, that the case $2j > a$ never occurs when Rule R2 applies, and that the inductive restart in Rule R2b is valid. In the arguments below, we assume that Bob has been able to apply the rules on all turns before the turn in question.

**Claim.** *If Rule R3 applies, then Bob is able to follow its instructions. Further, Rule R3 will apply on all of Bob's remaining turns.*

*Proof.* If $2j + 1$ has already been chosen, then since there are $n - (2j - 1) = 2m - 2j + 2 \geq 2$ remaining integers, there is at least one choice for $B_{2j}$ other than $2j$. After Bob's turn, Alice cannot choose $A_{2j+1} = 2j + 1$. Thus, Rule R3 applies on turn $2j + 2$, and by induction it applies on all of Bob's subsequent turns. □

It follows that if Rule R2 applies, then Rule R3 could not have previously been applied. Thus, after the most recent restart (if any), Rule R1b must have been applied on turn 2, and Rule R2a must have been applied on turn $2i$ for $2 \leq i < j$.

**Claim.** *If Rule R2 applies, then $2j \leq a$. Further, if Rule R2a applies, then Bob is able to choose $B_{2j} = 2j$. If Rule R2b applies, then $a$ is even, and all integers from 1 to $a$ are chosen on turns 1 to $a$.*

*Proof.* After Rule R1b was applied, 2 and $a$ had been chosen. Since Rule R2 has applied ever since, Alice has chosen all of the odd numbers from 3 to $2j - 1$, and since Rule R2a was applied on the previous turns, Bob has chosen all of the even numbers from 4 to $2j - 2$. Since $a$ is different from the other chosen numbers, and $a \neq 1$, we must have $a \geq 2j$. If Rule R2a applies, $2j \neq a$, so Bob can choose $B_{2j} = 2j$. If Rule R2b applies, then Alice chose $A_1 = a = 2j$ on her first turn, and Bob chooses $B_{2j} = 1$. Then all of the numbers from 1 to $a$ (and only those numbers) have been chosen. □

**Solution 2:** Let $S = \{k :$ the number $k$ was chosen on the $k$ th turn$\}$; at the beginning of the game, $S$ is empty, and each turn either adds an element to $S$ or keeps $S$ the same. Call a number "available" if it hasn't been chosen yet.

If $n$ is even and Bob chooses the "even" goal, then Bob can win by following the rules below on the $k$th turn (where $k$ is even).

(1) If all numbers greater than $k$ are available, then Bob chooses the one remaining available number in $\{1, \ldots, k\}$. This rule always applies when $k = n$.

(2) If $k \leq n - 2$ and Rule 1 doesn't apply, then Bob chooses $k + 1$ if available; if not, he chooses an available number greater than $k + 1$ if possible; otherwise, Bob can choose any remaining value other than $k$ (since $k < n$, there is more than one available number for Bob to choose).

**Claim.** *If $k$ is even, then after $k$ turns, $S$ has an even number of elements, and either* 0 *or at least* 2 *of the numbers greater than $k$ have been chosen.*

The proof is by induction on even values of $k$ from 0 to $n$, the base case $k = 0$ being the beginning of the game with 0 elements in $S$ and 0 numbers chosen. Assume now that the claim hold for some even $k < n$.

If $k = 0$ or if Bob applies Rule 1 on the $k$th turn, then all numbers from 1 to $k$ are chosen before Alice makes the $(k+1)$st turn. If Alice chooses $k+1$ (adding an element to $S$) or $k+2$, then Bob applies Rule 1 on the $(k+2)$nd turn to choose $k+2$ (adding an element to $S$) or $k+1$, keeping an even number of elements in $S$ and leaving 0 chosen numbers greater than $k+2$. If, on the other hand, Alice chooses a number greater than $k+2$, then exactly 1 number greater than $k+2$ has been chosen before Bob makes the $(k+2)$nd turn. Bob must then apply Rule 2, and since $k+2 \leq n-2$ in that case, there is still an available number greater than $k+2$ for him to choose. Thus, Alice's and Bob's turns add no elements to $S$, and result in 2 chosen numbers greater than $k+2$.

If $k > 0$ and Bob applies Rule 2 on the $k$th turn, then either Bob chooses $k+1$, or $k+1$ was previously chosen; either way, Alice can't add an element to $S$ on the $(k+1)$st turn. Since Rule 1 didn't apply on the $k$th turn, by the inductive hypothesis there were already at least 2 chosen numbers greater than $k$, and thus at least 1 chosen number greater than $k+1$. If before Bob makes the $(k+2)$nd turn, the only chosen number greater than $k+1$ is $k+2$, then Bob applies Rule 1 and chooses a number strictly less than $k+2$; this adds no new element to $S$, and leaves 0 chosen numbers greater than $k$. Otherwise, there is at least 1 chosen number greater than $k+2$ before Bob makes the $(k+2)$nd turn, so Bob applies Rule 2, which never adds an element to $S$. Either he chooses a number greater than $k+2$, making at least 2 such numbers chosen, or else all of the numbers (and in particular, at least 2 numbers) greater than $k+2$ were already chosen before Bob's turn.

This completes the induction that proves the claim. Applying the claim with $k = n$ shows that Bob wins when $n$ is even.

If $n$ is odd and Bob chooses the "odd" goal, then Bob can win by following the rules below on the $k$th turn; since $k$ is even, $k < n$.

(1) If $S$ has an odd number of elements, then Bob chooses $k+1$ if available; otherwise, Bob can choose any number other than $k$ (since $k < n$, Bob must have an option other than $k$).

(2) If $S$ has an even number of elements, then Bob chooses $k$ if available; otherwise, Bob chooses a number less than $k$ (since only $k-1$ numbers are chosen before Bob's turn, there must be an available number less than or equal to $k$).

If Bob applies Rule 1, then he doesn't add an element to $S$, and Alice can't add an element to $S$ on the $(k+1)$st turn, so the number of elements in $S$ remains odd after the $(k+1)st$ turn. By induction, Bob applies Rule 1 on all future turns, and the number of elements in $S$ remains the same for the rest of the game, so Bob wins.

Assume hereafter that Bob is never able to apply Rule 1 for the entire game.

**Claim.** *If $k$ is even, then after $k$ turns, either $S$ has an even number of elements and no numbers greater than $k$ have been chosen, or $S$ has an odd number of elements and exactly one number greater than $k$ has been chosen.*

In the base case $k = 0$, there are 0 elements in $S$ and no elements at all have been chosen. Proceeding by induction, assume that the claim holds for some even $k < n - 1$.

If after $k$ turns $S$ has an even number of elements, then by the inductive hypothesis all numbers less than or equal to $k$ have been chosen so far, and only those numbers. To prevent Bob from applying Rule 1, Alice must not choose $k+1$ on the $(k+1)$st turn. If Alice chooses $k+2$, Bob chooses $k+1$ on the $(k+2)nd$ turn; then $S$ still has an even number of elements and no numbers greater than $k+2$ have been chosen. If Alice chooses a number greater than $k+2$, Bob chooses $k+2$, adding an element to $S$; then $S$ has an odd number of elements, and exactly one number greater than $k+2$ has been chosen.

If after $k$ turns $S$ has an odd number of elements, then Alice must choose $k+1$ on the $(k+1)$st turn, adding an element to $S$ to prevent Bob from applying Rule 1. By the inductive hypothesis, exactly one other number greater than $k$ was chosen before Alice's turn, and we now know that number couldn't have been $k+1$. If that number is $k+2$, then all but one of the numbers from 1 to $k+2$ have been chosen prior to the $(k+2)$nd turn, and Bob chooses the remaining avalaible number in that range; then $S$ has an even number of elements, and no numbers greater than $k+2$ have been chosen. If, on the other hand, a number greater than $k+2$ was previously chosen, Bob chooses $k+2$, adding another element to $S$; then $S$ has an odd number of elements, and exactly on number greater than $k+2$ has been chosen.

This completes the induction that proves the claim. Applying the claim with $k = n - 1$, either $S$ has an even number of elements and Alice is forced to choose $n$ on the $n$th turn, adding an element to $S$, or $S$ has an odd number of elements and $n$ is already chosen, so Alice is unable to add an element to $S$ on the $n$th turn; either way, Bob wins.