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MAA INSTRUCTIONAL PRACTICES GUIDE

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Manifesto: A declaration of values

Success in mathematics opens opportunities for students. A wealth of research literature exists on how mathematics instructors can facilitate rich, meaningful learning experiences and on what instructors can do to improve teaching and learning at the undergraduate level: Effective teaching and deep learning require student engagement with content both inside and outside the classroom. This Instructional Practices Guide aims to share effective, evidence-based practices instructors can use to facilitate meaningful learning for students of mathematics. Professional associations in the mathematical sciences along with state and national funding agencies are supporting efforts to radically transform the undergraduate education experience; it is truly an exciting time to be a mathematics instructor!

With that big picture in mind, this guide is written from the perspective that teaching and learning are forces for social change. Beyond the confines of individual instructors' classrooms, beyond their decisions about what mathematics to teach and how to teach it, there are societal forces that call upon all mathematics instructors to advocate for increased student access to the discipline of mathematics. Inequity exists in many facets of our society, including within the teaching and learning of mathematics. Because access to success in mathematics is not distributed fairly, the opportunities that accompany success in mathematics are also not distributed fairly. We in the mathematical sciences community should not affirm this inequitable situation as an acceptable status quo. We owe it to our discipline, to ourselves, and to society to disseminate mathematical knowledge in ways that increase individuals' access to the opportunities that come with mathematical understanding.

Some of us have become reflective instructors over the course of our careers, and our classrooms have changed and improved as a result. But if we truly want to effect change, then we are compelled to extend the reach of our efforts beyond our own students in our own classrooms. It is our responsibility to examine the system within which we educate students and find ways to improve that system. It is our responsibility to help our colleagues improve and to collectively succeed at teaching mathematics to all students so that our discipline realizes its full potential as a subject of beauty, of truth, and of empowerment for all.

Such a sea change will require transforming how mathematics is taught and facing our own individual and collective roles in a system that does not serve all students well. Societal norms tend toward a belief that only a certain kind of individual can do mathematics and other kinds of people need not even try. We in the profession of teaching mathematics must look inward to determine if we are doing our part to dispel this myth.

All instructors can facilitate student success in mathematics, and we cannot underestimate the power of the environment in our classrooms, departments, and institutions to positively impact student learning. Changing teaching practice is hard. But those of us who do mathematics recognize the hard work required to learn and understand it and we choose to do that hard work. We can likewise choose to do the hard work required to teach our beloved subject.

Mathematics instructors stand at a crossroads. We must gather the courage to take the difficult path of change. We must gather the courage to venture down the path of uncertainty and try new evidence-based strategies that actively engage students in the learning experience. We must gather the courage to advocate beyond our own classroom for student-centered instructional strategies that promote equitable access to mathematics for all students. We stand at a crossroads, and we must choose the path of transformation in order to fulfill our professional responsibility to our students. This Instructional Practices Guide can serve as a catalyst for community-wide transformation toward improved learning experiences and equitable access to mathematics for all students. Society deserves nothing less.

Introduction to this guide

The most recent MAA documents, *Committee on the Undergraduate Programs in Mathematics (CUPM) Curriculum Guide to Majors in the Mathematical Science* (Zorn, 2015) and *A Common Vision for Undergraduate Mathematical Sciences Programs in 2025* (Saxe and Braddy, 2016) serve as an impetus for this *Instructional Practices Guide*. The *CUPM Curriculum Guide* provides course recommendations along with sample syllabi for mathematical science courses, but does not provide specific teaching strategies faculty have found to be effective with their students, and *Common Vision* calls for the use of evidence-based instructional strategies by reiterating the call from the INGenIOuS project report (Zorn, et al., 2014):

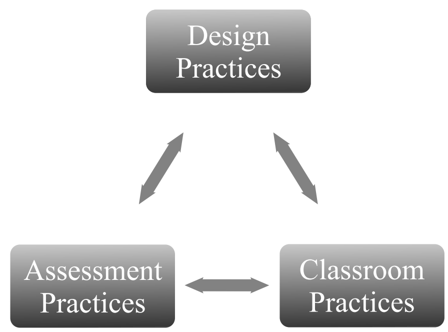
> We acknowledge that changing established practices can be difficult and painful. Changing cultures of departments, institutions, and organizations can be even harder. But there is reason for optimism. In mathematical sciences research, we are always willing, even eager, to replace mediocre or "somewhat successful" strategies with better ones. In that open-minded spirit, we invite the mathematical sciences community to view this call to action as a promising opportunity to live up to our professional responsibilities by improving workforce preparation (p. 25).

With this in mind, this *Instructional Practices Guide* is designed as a "how to" guide focused on mathematics instruction at the undergraduate level. It is based on the concept that effective teaching is supported by three foundational types of practices: **classroom practices, assessment practices,** and **course design practices** all informed by empirical research as well as the literature on technology and equity. In this introduction, we describe the intended audience, provide a brief overview of each practice, and offer suggestions on how to navigate this guide.

The *Instructional Practices Guide* is founded on the belief that every student should have the opportunity to engage in deep mathematics learning, guided and mentored by their instructor. It is intended for all instructors of mathematics, from the new graduate teaching assistant to the most experienced senior instructor; from the contingent faculty member at a two-year institution to the new faculty member at a doctoral-granting institution; from the instructor who wants to transform her own teaching to the mathematician or mathematics educator facilitating professional development for graduate students or collegiate faculty. It is also intended for administrators who are in positions to work with their faculty to initiate systemic change in their departments and across their institutions. Administrators will recognize that many of our suggestions are applicable to other disciplines; in fact, some of the suggestions are borrowed from research in science education

The **Classroom Practices** chapter provides examples of teaching practices, both inside and outside the classroom, that foster student engagement as well as a section on selecting appropriate mathematical tasks that contribute to building a sense of community within the classroom. The **Assessment Practices** chapter builds on policy assessment documents from various associations including the National Council of Teachers of Mathematics, the American Statistical Association, and, of course, the MAA. This chapter centers on the interplay between formative and summative assessment to examine the teaching and learning of mathematics with a strong focus on learning outcomes. The **Design Practices** chapter provides the reader a brief introduction to instructional designs that help achieve desired learning outcomes, based on theories of design, along with potential challenges and opportunities associated with instructional design.

We acknowledge the suggestions in this guide are not exhaustive, but we aim to include something of interest for any reader to adapt for their own classroom. Each of the practices informs the others, and depending on readers' experiences, they might choose to read the guide in an order other than the one presented. We purposefully begin with the Classroom Practice chapter in an effort to engage readers who are just beginning to transform their teaching. As readers gain more experience with student-centered teaching practices, they can navigate back and forth among the chapters as needed. For example, a reader more experienced with student-centered teaching might begin by reading the Design Practices chapter to prepare for designing a new course, then read the Classroom Practices chapter to prepare specific lessons and activities, then read the Assessment Practices chapter to garner formative assessment strategies, redesign a lesson or classroom activities based on the results of the formative assessment, and then learn about novel summative assessments. The model shown below indicates the fluid way in which readers might utilize the guide.



We also acknowledge transforming one's classroom practices takes time, and we firmly believe specific examples are helpful in facilitating such a transformation. Throughout the guide we offer vignettes that are both easy to follow and informed by the substantial body of research regarding effective teaching and deep student learning.

The crucial finding from the research upon which this guide is founded is that effective teaching and deep learning require student engagement with mathematics both inside and outside the classroom. Bringing student ideas, beliefs, and practices into the direct view of peers and instructors enriches teaching and learning and promotes community in remarkable ways. The vast body of evidence strongly supports the transformational power of these practices in prompting changes in instructors and students at all levels from all demographic backgrounds.

Indeed, such transformation can promote diversity, inclusion, cultural responsiveness, and social justice within the mathematical sciences community. Our task as a community is to create these meaningful and inspiring mathematical experiences for all our instructors and students. As such, we conclude the document with a brief discussion on cross-cutting themes regarding technology and equity, two important topics that are intertwined in each of the other chapters. We strongly encourage our readers to reflect on how they integrate technology into each of the practices and how their practices promote equity in the mathematics classroom.

In summary, this *Instructional Practices Guide* is a call to the mathematical sciences community to scale up the use of evidence-based instructional strategies and to collectively and individually hold ourselves accountable as professional educators for improving the learning experiences of all undergraduate mathematics students.

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Contents

Classroom Practices

CP.1. Fostering student engagement

The purpose of this chapter is to provide college and university instructors with accessible starting points for implementing practices that foster student engagement. Classroom practices aimed at fostering student engagement attend to the research-based idea that students learn best when they are engaged in their learning (e.g., Freeman, et al., 2014). Consistent use of active learning strategies in the classroom also provide a pathway for more equitable learning outcomes for students with demographic characteristics who have been historically underrepresented in science, technology, engineering, and mathematics (STEM) fields (e.g., Laursen, Hassi, Kogan, and Weston, 2014). In this section we focus on classroom practices that enable students to be actively engaged in their own learning. We illustrate what it means to be actively engaged in learning and we offer suggestions to foster student engagement.

Student engagement can be enhanced by activities that require sense-making, analysis, or synthesis of ideas during class. These strategies may be "anything course-related that all students in a class session are called upon to do other than simply watching, listening and taking notes" (Felder and Brent, 2009, p. 2). Such practices may be new to students, thus we caution instructors to be mindful of students' interactions and responsiveness to these new teaching techniques. Historically, students' learning experiences of mathematics generally involve memorization and rote repetition. As such, in transforming one's teaching instructors may need to provide structures for peer-to-peer communication and display a genuine interest in student contributions in the classroom that moves beyond questioning solely dependent on memorization and rote applications. Instructors will need to create a classroom environment where students feel accountable both as individuals and as members of the classroom community of learners.

Thus, we begin by providing suggestions on how to build a classroom community and then turn to quick-to-implement strategies (e.g., wait time after questioning and one-minute papers), followed by more elaborate strategies that may require more preparation (e.g., collaborative learning strategies, flipped classroom, just-in-time teaching). While many of our suggestions focus on in-class activities, out-of-class activities are also important and can impact the classroom dynamics, discourse, and community (Pengelley, 2017).

CP.1.1. Building a classroom community

The connection between student success and student engagement is supported by several studies (e.g., Freeman, et al., 2014; Hake, 1998; Kuh, 2007). In fact, the Community College Survey of Student Engagements (CCSSE) "use of student engagement as a proxy for student academic achievement and persistence" has been validated by results from three recent studies (McClenney, Marti, and Adkins, 2007). However, national CCSSE data underscore the need for a more deliberate and immediate approach to establishing a classroom community that supports engagement. In 2016, the CCSSE showed low rates of student connections with instructors, other students, and institutional resources. Of the students surveyed in the 2016 CCSSE, only about 50% report discussing grades or assignments with their instructor or working with other students during class. The National Survey of Student Engagement (NSSE) shows similar results.

Whereas community and sense of belonging are more likely to flourish in classrooms where the instructor incorporates student-centered learning approaches (see Slavin, 1996; Rendon, 1994), establishing norms for active engagement or taking steps to increase a student's sense of belonging to the classroom community also impacts the quality of student engagement in the classroom. Thus, setting the classroom norms and engagement expectations to incorporate student-centered learning on the first day of class is important. Such norms can be related to in and out of classroom settings. For example, taking time to build a classroom community allows students to form and broaden their support networks outside of class which can include other students, instructors, or other campus resources. Another benefit of such networks is that they play an important role in the success of members of marginalized communities (Leyva, 2016; Treisman, 1992).

Classroom vignette: First day

To establish a personal connection with her students and to create a classroom culture of participation, Dr. Garcia makes it a priority to learn students' names and something unique about them on the first day. This helps Dr. Garcia build community, establish rapport with her students, and give them a sense of the class structure.

After briefly reviewing the contents of the course syllabus, she passes out large note cards where students write the name they prefer to be called on both sides of the card in print large enough to be read from anywhere in the room. Sometimes she takes pictures of students with their note cards to review for the next class. In an effort to get to know her students better and to encourage class interaction, students write their name, most recent mathematics course, intended major, and three interesting facts about themselves on 3 by 5 cards. Students then pair up and introduce each other to the class using one of these facts. It is important that her students also get to know her and thus Dr. Garcia also shares three interesting facts about herself. She may say, "You wouldn't know it by looking at me but…." She often finds that students feel more comfortable participating because they are able to perceive her as their instructor, as a person, and as a learner like them.

After the initial work of learning names and sharing interesting facts, Dr. Garcia's students work on a carefully selected mathematical task (see section CP.2). She monitors the discussions and follows this with a class discussion about prior learning experiences where each group shares one or two techniques that they think would help with their learning. She uses students' work on the task to ground the discussion and keeps the students' techniques in mind for future teaching strategies. Students leave the first class with a homework syllabus quiz and a campus resource task, which encourages students to read the syllabus and explore campus resources. A recurring question on the syllabus quiz asks students to report the name and contact information for at least three other students in the class.

On the second day of class, Dr. Garcia requests that everyone have their name cards out. She reminds them that it very important for her to learn as many names as possible, and that her goal is to know all names within the first few weeks. Next, she introduces proposed norms for the class, based on ideas she heard from students in the first class session. Dr. Garcia explicitly attributes norms to what she heard and thus sends the message that students' ideas are important.

Discussion. From the first moment of a class, Dr. Garcia intentionally connects everyone in the classroom. To enforce the notion that Dr. Garcia is approachable, she chooses to share personal anecdotes or interesting facts about herself. Having students share with one another on the first day of class also helps students better connect with others in the classroom and emphasizes the type of interactions that will be used in the classroom. Having students spend time to establish connections with each other and understand the resources available to them helps build their learning community both inside and outside of the classroom.

Practical tips

When requiring more interaction in the classroom, it is helpful to establish behavioral norms and guidelines for productive exchanges. Many colleges have established principles on developing community among students (e.g., Valencia College publishes a set of principles on how members of their college community treat one another, http://valenciacollege.edu/PJI/principles.cfm) and promoting cooperative behavior in the classroom. To this end, it is important to have a conversation about the expected behaviors. For example, late arrivals to class impact all group members and unnecessary cell phone use unfairly distracts from group interactions and attentiveness in class, but the willingness to listen intently and communicate ideas about the mathematics promotes learning and engagement. The following are more ideas for creating a classroom community.

1. Promote a friendly atmosphere while attending to students who are members of marginalized communities.
2. Establish the importance of arriving prepared for class by completing assigned readings, videos, or homework before coming to class.
3. Model reaching consensus to arrive at resolutions to questions posed to the class.
4. Remind students that when working in groups it is important to listen carefully and with respect. This includes listening with the intent to understand others' strategies and questions and not dominating group conversations or classroom interactions.
5. Focus on the mathematics and critique ideas, but do not criticize people.
6. Help students take responsibility for their own learning by asking them to share strategies and questions with the goal of communicating their reasoning instead of using unhelpful phrases such as "I just don't get…."

CP.1.2. Wait time

Questioning techniques cannot meaningfully foster student engagement if students are not given enough time to think about the questions posed and to respond. The time instructors allow for this is called wait time. Specifically, wait time refers to (1) how long the instructor waits for students to consider and respond to a question or (2) how long the instructor waits for a student who pauses during or after their response.

Research shows that on average instructors wait less than 1.5 seconds before they either answer their own question or ask a follow-up question. This practice can result in lowering the cognitive demand of tasks (Tobin, 1987) and discouraging students' deep engagement in mathematics. It also communicates to students that a response is not actually necessary or that they are not expected to answer questions. To encourage student participation, research points to the need for instructors to wait at least seven seconds and that an average wait time greater than three seconds is a threshold for changing instructor-student discourse (Fuller, et al., 1985; Tobin, 1987). Although seven seconds may not seem like a long time, when enacted it can feel like an eternity, especially when students are not used to the instructor genuinely expecting them to think about and respond to their questions. Benefits of allowing students enough time to think and respond include: a decrease in the number of "I don't know" responses; an increase in the number of students that respond to questions; more correct or more sophisticated responses; and greater conceptual depth of student responses.

Classroom vignette: Wait time

In this example from a calculus course, Dr. Smith's goal is to develop the integral formula for work done by a non-constant force, $W = \int_a^b F(x)\,dx.$ In particular, Dr. Smith discusses the work done to compress a spring by 0.2 meters using Hooke's law, which says $F(x)=kx$.

Dr. Smith: Why can't we just use $W=F \cdot d$ with $d=0.2$ meters? [waits 1.5 seconds]

Dr. Smith: Because the force is non-constant, right? So why does that matter? [waits less than 1 second]

Dr. Smith: Because $W=F \cdot d$ only works when both quantities are constant. So then what should we do instead? [waits 1.5 seconds]

Dr. Smith: We should slice the total compression distance into small pieces of length Δx. Why does that help? [waits 2 seconds]

Dr. Smith: Because on those slices, the force is approximately constant. So then how can we compute work? [waits 1 second]

Dr. Smith: Well, then we can use $W=F \cdot d$. But what's d for these slices? [waits less than 1 second]

Dr. Smith: Wouldn't it be $d=\Delta x$? Why? [waits 1 second]

Dr. Smith: Because the slices are of length Δx. So, how much work is done on each slice? [waits 1.7 seconds] Donna?

Student: I don't know.

Dr. Smith: That's just $F(x) \cdot \Delta x$, so $kx\Delta x$.

Discussion. This vignette illustrates how this instructor's use of wait time likely did not allow students to reason about and engage in the ideas. We invite the reader to imagine a situation in which a longer wait time occurs. In such a situation, an increase in wait time not only benefits student engagement in mathematics, but it can also result in positive changes in instructor questioning. For example, as instructors increase their wait time, they tend to decrease the number of questions that require only a quick, factual, or procedural response and increase the quality and variety of questions asked. It is important to acknowledge that better questioning and more wait time for student responses will increase the time needed per topic or lesson. However, as the quality and variety of questions asked increases, so does students' higher order thinking and engagement. More on instructor-questioning strategies can be found in section CP.1.3 and section CP.1.5.

Practical Tips

- Remember that wait time is important in creating a sense of community in the classroom and students might not be accustomed to long wait times. Remind students that the wait time is important so they have time to think and respond.
- Ask questions such as, "Do you need me to repeat the question?" or "Do you want me to rephrase the question?"
- After asking a question, actually keep track of how many seconds you wait. Some instructors count to seven on their fingers behind their back.
- Tell students why you are waiting and the benefits for them when you wait.
- If you reach 10 or more seconds with no response, consider making use of the Think-Pair-Share strategy discussed in section CP.1.5 of this chapter.
- For particularly challenging questions, consider revisiting this question as a one-minute paper prompt. See section CP.1.4 of this chapter.

For more tips on integrating wait time effectively in the classroom see www.ericdigests.org/1995-1/think.htm.

CP.1.3. Responding to student contributions in the classroom

Instructors often greet students' excellent ideas with enthusiasm, praise, and a positive facial expression. When students see a less-than-enthusiastic look on an instructor's face, they may start to backpedal with their response. This is especially true when students are appropriately challenged, where they experience struggle and sometimes failure. On the other hand, research shows that productive struggle can help students develop persistence and confidence (Edwards and Beattie, 2016) which can lead to successful learning. In this section we discuss how to respond to incorrect answers, incomplete explanations, faulty arguments, and students' struggles.

From an equity stance, one of the most powerful ways an instructor can build community and student confidence is to reframe errors. Instead of simply listening for and identifying student errors, learn to listen for aspects of correct reasoning in students' responses. Instead of viewing struggle as a problem, learn to say to the class "I love that we are struggling with this, it is an opportunity to learn!" This communicates to the students that the instructor values the students' reasoning and that she supports students as they move from their current understanding towards coherent understanding.

To help facilitate a classroom environment where questioning and justification are normative, instructors need strategies for responding to student errors in classwork and class discussions. The following vignettes contain ideas on how instructors may respond to students that will facilitate the development of a community in the classroom and equip instructors with strategies to manage student responses (e.g., Battey and Stark, 2009; Hughes, 1973).

Classroom vignettes: Responding to students

For each vignette below, there are several possible responses. You may want to consider your own response before reading the options provided.

Vignette 1: A group of students are working a problem together at the board and Dr. Bird hears one of them make a strong (correct or incorrect) assertion. None of the other students question the assertion. Dr. Bird usually opts to use one of these three interventions.

1. She works with just that group and asks the student to revisit that assertion and explain their thoughts out loud. If the assertion is incorrect, she waits to see if the student self-corrects, if another student makes a suggestion, or if she can ask a follow-up question that helps the student recognize the incorrect assertion. If the assertion is correct, she waits to see if students in the group can defend the assertion, answer probing questions related to the assertion, and that they are not just accepting the assertion due to the dominance of the group member or for other reasons.
2. She regroups the class and shares the assertion. She then uses some form of think-pair-share (see section CP.1.5) to help students examine the assertion for what makes sense and what needs further reasoning.
3. She introduces the assertion on a homework problem or assignment as an example of student reasoning and asks the students to explain why the assertion is correct or incorrect (and perhaps also what related idea is correct). Alternatively, she presents the assertion on the homework and asks students to decide whether the assertion is true and to justify their answer.

Vignette 2: In the middle of a lecture Dr. Brown poses a question to the class. A student who rarely speaks up volunteers an incorrect answer.

1. Dr. Brown responds by asking the student to explain their reasoning. After the student has explained their reasoning, Dr. Brown poses a question to help clarify, offers a counterexample, or validates any correct reasoning while offering "another way to reason about his original question" that helps bridge the student response with the expected one.

2. If Dr. Brown is familiar with the misconception, he may say "yes, it's really common or tempting to think about it that way, but here's an example that doesn't fit the pattern." He then offers, "It may be more helpful to think about the concept…." Many times, he validates the student's response by thanking them for bringing up the idea and that he intended to mention the pitfall or tempting misconception. See also the vignette following section CP.1.9.
3. Dr. Brown may pose the answer as a "conjecture" and ask students to work in pairs (e.g., section CP.1.5) and reason about the conjecture.

Discussion. Appropriately handling student responses whether correct, incorrect, or unexpected and embracing student contributions in the classroom directly affects the learning environment and can encourage student-centered learning in the classroom. To encourage student responses and participation, it is important to recognize the value of students offering both correct and incorrect responses. If Dr. Brown only engages groups who make incorrect assertions, this implicitly communicates that there may be something incorrect with their mathematical reasoning. Instead, Dr. Brown established classroom norms for which following up with individual students and groups is not interpreted as a sign of being incorrect, but rather part of the classroom participation structures.

As instructors choose approaches, it is important to be intentional about the purpose of the questions as well as the questioning techniques. Determine if the purpose of the question is to do a quick check for learning and retention, to assess prior knowledge, or to elicit discussion. The purpose of the question can inform the way in which you handle any responses, correct or incorrect.

Practical tips

These tips first appeared in the MAA Teaching Tidbits Blog, maateachingtidbits.blogspot.com.

1. Create a safe space for incorrect answers. This takes time and care. For example, you can say "I'm so glad you raised that point. We often think [incorrect idea] because [some kind of reason], but actually if you take into account [key idea] it leads to this other way of reasoning, which is correct." This emphasizes that reasonable attempts at solving a problem can sometimes lead to incorrect solutions.
2. Keep a poker face. Make sure no matter what the student says that you ask the student to justify the reasoning behind the answer. Try to not give away whether the answer is correct. Another option is to have a different student discuss whether the answer is correct or incorrect and explain why.
3. Focus on the reasoning. The poker face is also important to encourage students to share their reasoning, without fear of discouragement from negative reactions. It also prevents them from changing their answer (based on the look on your face) without diagnosing the cause of their error.
4. Distinguish between types of errors. You may or may not want to give a lot of time to discussing a typo, versus a common misconception or confusion. Sometimes it is important just to correct and move on.
5. Identify correct aspects of a solution. Even though a solution may be incorrect, the student may have done some good work to get there. In some cases you can say, "That would be the correct answer if [xxx], but actually we are thinking about [yyy]."
6. Keep in mind that speaking in front of peers and the instructor is risky. A way to lessen the pressure is to give students the opportunity to come back to their idea. This could be as simple as asking a student to rethink an assertion, and say "Gloria, shall we come back to you? Does that sound good? I think we all want to know what you are thinking, so let's hear from someone else and come back to see what you are thinking."

CP.1.4. One-minute paper or exit tickets

Instructors use one-minute papers or exit tickets to quickly assess what students learned from a class session or their general thoughts about the course. The use of one-minute papers and exit tickets may enhance student engagement because students are required to reflect upon the learning taking place, to demonstrate a skill, or to communicate a concept at the close of a topic or class session.

Not surprisingly, a one-minute paper takes about a minute to complete and is usually integrated at the end of class, although it can also be used at the end of a section or topic. The instructor poses a question that prompts students to reflect upon significant concepts a student learned that day or on concepts for which they still feel uneasy or did not understand. Students address the question in writing and hand it in, but they can also complete their one-minute paper via an online learning management system and submit it by the end of the day. Both mechanisms allow instructors to review submissions quickly and obtain formative feedback about student learning.

Similarly, exit tickets function like one-minute papers except that they may consist of a short-answer question or a multiple-choice question that students must answer by the end of a class session and submit as they exit the classroom. Again, this can serve to determine how well students understand new material.

Classroom vignette: One-minute exit ticket

Dr. Kessler introduced Taylor series during a class meeting of his calculus course. He wants to assess what students learned from the class session as well as what they felt they did not fully grasp in order to address these points in the next class session. Thus, at the end of the class period he asks students to take one minute to explain in concise, complete sentences:

1. What are the three most significant things you learned today about Taylor series?
2. What are you left wondering about Taylor series?
3. Is there anything that still is unclear about Taylor series for you?
4. Why are you studying Taylor series?

He collects the one-minute papers and reviews them before the next class. He uses responses to the first question to determine how much students said they learned, responses to the second question to connect student "wonderings" to the next class session, and responses to the third question to construct follow up in-class or homework tasks. While it is not always necessary to request student names on one-minute papers, Dr. Kessler requests that students write their names on the paper, so he can plan individualized follow up with students, if necessary.

Practical tips

One-minute papers and exit tickets do not take a significant amount of time to review, and they provide important information on the status of student learning in the classroom. The one-minute papers can also be used to obtain general feedback about the course by having students reflect, once or twice during the semester, on "What is going well in this class?", "What needs to be modified?", and "What should be maintained?" An exit ticket may be used to assess how well student groups are functioning by asking students to respond to questions such as "Does working in this group enhance my learning?" or "Does working in this group hinder my learning?" A one-minute paper could be used for the same purpose by asking students to "Explain how working in this group enhances your learning" or "Explain how working in this group does not enhance your learning." Note that in this case the exit ticket questions are short questions that cannot be answered "yes" or "no". Exit tickets can also incorporate a mathematical task or question to formatively assess students' knowledge on a particular mathematical concept.

It is important that one-minute papers and exit tickets

- take less than one minute to address. However, some implementations of one-minute papers allow students more than one minute but less than about five minutes to complete the task.
- contain clear directions. The questions can be standard such as, "What is the most significant concept you learned today?", "What do you still wonder about?", and "What do you still not understand?" If several questions are posed, it helps to have them written on the board or projected on a screen.

CP.1.5. Collaborative learning strategies

Collaborative learning and cooperative learning are terms often used interchangeably, but the meanings of the terms differ. **Collaborative learning** typically refers to learning that takes place as small groups of students focus on open-ended, complex tasks, whereas **cooperative learning** typically refers to more structured, small-group learning that focuses on foundational or traditional knowledge with group roles (e.g., facilitator, summarizer, recorder, presenter) that may also serve to help students learn to work in groups (see Cooper and Robinson, 1997; Smith and MacGregor, 1992).

Johnson and Johnson (1999) indicate five basic elements essential for successful cooperative learning:

- **Positive interdependence**: Group interaction is necessary for successful resolution of the question or task and for linking individual success and the success of the group. For example, a task can be broken into parts to be completed by individuals but the individual work is needed for a group resolution. The relationship between individual and group success is exemplified by the fact that students work on a task together, but submit one group response orally or in writing
- **Face-to-face interaction**: Group interactions include discussing solution paths, important concepts, connections to prior knowledge, and facilitating words of encouragement and help when needed. For example, when a student asks the instructor, "Is this right?" the instructor can redirect the question to the group and ask for input from others in an effort to help the students answer their own questions.
- **Individual accountability**: Students are held accountable for their share of the work in the group. For example, a portion of a student's grade for group work may depend on an individual quiz given at the end of the activity, or there may be questions in a task that must be answered individually.
- **Social skills**: Group interaction requires interpersonal, social, and collaborative skills. Instructors must provide students with guidance on how to effectively interact in a small group. For example, a class discussion of appropriate group behaviors and expected norms of communication is an essential precursor to implementing successful cooperative learning. Providing a handout listing these behaviors and reminding students of these expectations throughout the term is also important.
- **Group processing**: Group members discuss effectiveness in reaching their goals and in working together. For example, students should be given time to reflect on prompts such as, "What I liked most about this group was…" or "Our effectiveness as a group could be improved by…," and then the students should discuss their responses in the group. It also may be helpful for the instructor to collect these reflections.

Implementing collaborative or cooperative learning strategies successfully relies heavily on assigning groups of an appropriate size and with students and task in mind. For some strategies (e.g., Think-Pair-Share) pairing students in groups of two by proximity is appropriate. For other strategies (e.g., Small Group Work), a group size of no more than three students is ideal because, by nature of the size of the group, all students have more opportunities to contribute to group discussions than they would in a larger group in which less assertive students may not have ample opportunities to contribute. Sometimes it is appropriate to randomly assign groups and other times it is appropriate to assign groups by thinking carefully about the task to be completed and the skills each student brings to the task. For example, in a class of 40 students facilitated by an instructor and a graduate student, providing students with a diagram of the classroom set-up (i.e., location of groups 1-13) and pre-assigning groups so that students know where to go at the beginning of class is preferable to an ad-hoc approach of having students count off from 1 to 13 and then "find" their groups. The latter also does not allow for placing students in groups based upon performance on homework or other factors that may influence group dynamics.

In the following sections, we discuss specific cooperative learning strategies, such as think-pair-share, paired board work, and small group learning.

Think-pair-share

Think-pair-share is a cooperative learning strategy that requires students to think about a question, discuss their thinking with a partner, and then verbally share their ideas in class or submit their ideas for review. Brame and Biel (2015) describe think-pair-share as follows: "The instructor asks a discussion question. Students are instructed to think or write about an answer to the question before turning to a peer to discuss their responses. Groups then share their responses with the class." Novices to collaborative learning may find that the think-pair-share cooperative learning strategy is a good first step toward implementing cooperative learning in the classroom.

Classroom vignette: Think-pair-share

Dr. Adams attempts to make his lectures interactive and to motivate concepts with as much student participation as he can garner in a lecture course attended by 90 students. Dr. Adams motivates the Extreme Value Theorem in calculus using think-pair-share several times via a cycle of questioning, individual time to think, sharing with a partner, and reporting out; then he repeats the process with follow-up questions.

Dr. Adams asks each student to take out a sheet of paper and graph a function over the interval $[a, b]$ and to pass their paper to someone in front of or behind them. He then asks them to pair with a student beside them, and he begins to pose questions about the graphs that they have at hand. Given that the task is relatively open-ended, the expectation is that there will be many different graphs generated by the students. After each question he gives students time to think about their answer and to discuss with their partner. He poses questions such as, "Does your function have a maximum value over the interval $[a, b]$?" "Does your function have a minimum value over the interval $[a, b]$?" and "Does your function have both a maximum and minimum value over the interval $[a, b]$?" He then polls the class by asking students to "Raise your hand if the function you have has a maximum value over the interval $[a, b]$," "Raise your hand if the function you have has a minimum value over the interval $[a, b]$," and "Raise your hand if your function has both a maximum and minimum value over the interval $[a, b]$." After each question and poll, he asks students to discuss their answer with their partner. Students who raise their hand to the question, "Raise your hand if your function has both a maximum and minimum value over the interval $[a, b]$" share their functions by sketching them on the board or portraying them via the document camera. Recall that these are graphs of functions drawn by classmates.

After this Dr. Adams asks students to think (individually) about any features that the displayed graphs have in common and to discuss their thoughts with their partner. He asks students to share their findings by calling on student pairs. He uses the ideas presented by the students regarding the common features present to pose additional questions such as, "Is this condition really necessary?" and "What restrictions or guidelines would we need in order to guarantee that a function has a maximum and minimum value on the interval?" Dr. Adams has witnessed how students' engagement in creating their own functions and examples results in them feeling a sense of satisfaction when they establish that requiring a function to be continuous on $[a, b]$ will guarantee that the function attains an absolute maximum and absolute minimum on the interval. In the process, he also exposes the logical implications (via their generated examples) that a function can attain an absolute maximum and absolute minimum on the interval $[a, b]$ and not be continuous, but that continuity guarantees that maximum and minimum values exist. The polling of the class also introduces an empirical element to the process of establishing the theorem.

Discussion. Some practitioners may argue that a pure form of think-pair-share involves students sharing their thinking with the class or writing down their thoughts and handing in their written response. However, in a lecture-based setting, especially one with over 60 students, as in the vignette, it is not practical to call on every student. In this case, the instructor may solicit answers that are qualitatively different from those already mentioned by other students.

The think-pair-share strategy gives students the opportunity to polish their mathematical reasoning and communication prior to presenting their ideas to the whole class. It is also particularly helpful for English language learners and students with learning disabilities to prepare their contributions for whole-class discussion with a partner. This allows time to process their reasoning and to practice their communication skills in a less high-stakes context.

In addition, the incorporation of think-pair-share strategies advances participation structures in undergraduate mathematics classrooms that depart from strictly teacher-student forms of communication. Encouraging students to share their mathematical reasoning with a partner is another mechanism to build a stronger sense of classroom community, as discussed earlier in the chapter.

Practical tips

It is helpful to think through the goal of your think-pair-share activity.

- Do you plan to use student reasoning to resolve the question in a meaningful way? If so, can you anticipate some of the responses? What follow-up questions might be needed?
- Does your question or task require students to speak to one another, to generate new ideas, or to stimulate diverse strategies or examples? How will you sequence the interaction? How long will you give students to think; pair; then share?
- Do you believe that incorporating the activity will enhance student reasoning? If so, in what way?

Paired board work

Another collaborative learning strategy that is effective in promoting classroom engagement is called **paired board work**. To help facilitate learning, students engage in mathematics through problem solving, making conjectures, discovering patterns, exploring informally and formally, and formalizing ideas, while having the opportunity to learn from their peers. Paired board work necessitates that all students demonstrate their knowledge—in pairs or even in triples—at the blackboards or whiteboards in a classroom.

The logistics of using paired board work during class is rather simple. First, it is important to have access to stationary (wall-mounted) whiteboards or blackboards. Ideally, all student pairs should have space at the boards, but if this is not possible, it is appropriate to have half of the students at the board and half at their desks. To implement the strategy, assign students to work in pairs at the board with one student tasked as the scribe and the other student tasked as the quality controller. The scribe is responsible for writing the mathematics on the board. The quality controller is responsible for assisting the scribe and monitoring the quality of the mathematics displayed, attending to precision of notation, correctness, and accuracy. After each problem, students rotate roles so that each student has multiple opportunities to serve as both scribe and quality controller.

This method allows students to share their reasoning publicly, while also allowing instructors to formatively assess students' knowledge and skills. Often, students are asked to describe and present their reasoning of the work they have displayed on the boards and other students are asked to critique the presented reason-

Figure 1. Paired board work with calculus students.

Figure 2. Paired board work with college algebra students.

ing. It is important to remind students that we are critiquing the mathematics, not the person. Paired board work is a powerful tool for demonstrating knowledge, critiquing reasoning, analyzing multiple solution pathways, and assessing students' reasoning. Students enjoy having the opportunity to gain other students' perspectives on problems and discover other methods for completing various mathematical tasks. Figure 1 and Figure 2 below show examples of paired-board work in action.

There are many ways to assign tasks for students to complete at the boards when implementing paired board work. One example is to have all students work on the same problem at the same time. An advantage to this approach is that the instructor can assess students' thinking and abilities on this one problem in real-time, thus allowing the instructor to tailor their guidance for each pair as the instructor roams the room and monitors student progress. This approach also allows for multiple solution pathways to emerge, which can be leveraged by the instructor during the whole class debriefing stage before moving on to the next problem. An example of this is shown below in Figure 3, where different student pairs created different, and mathematically sound, solution pathways.

Maximum Rectangle Task: A rectangle is to be inscribed within a right triangle with a base of 3 and a height of 4. What is the largest rectangle that can be created?

Another approach when implementing paired board work is to have students work on different problems.

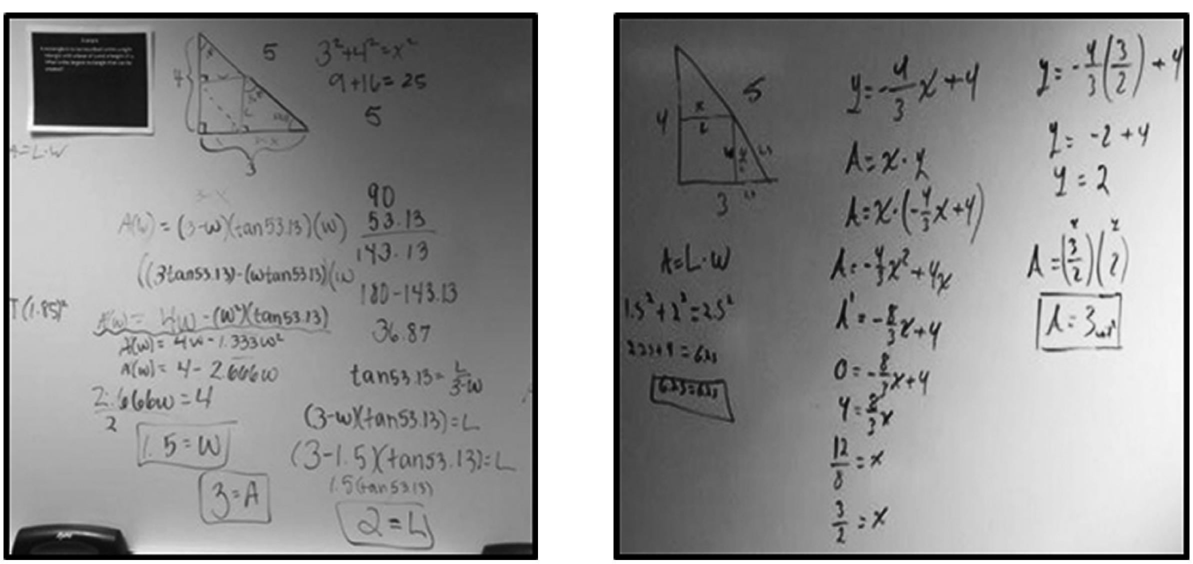


Figure 3. Multiple solution pathways of the Maximum Rectangle Task during paired-board work.

This approach can be accomplished by posting printouts of the problems at various places on the boards, and asking students to choose a problem to solve at the board. Once the task is completed, students can be called upon to present their solutions to the class and solicit feedback from their peers for improvement of their solutions. Overall, paired board work allows students to engage in mathematics through collaboration, to increase high-quality mathematical discourse, to critique the reasoning of others, to justify their solutions, and to display their reasoning publicly through written and verbal descriptions.

Small group work

Cooperative or collaborative learning must involve small groups, and there are many ways to incorporate small group work into the classroom. If grouping students based on performance, it is important to place low-performing students with medium-performing students and medium-performing students with high-performing students. This practice provides the best opportunity for students to work together and grow in their learning. It is best to avoid placing low-performing students with high-performing ones. The following is a list of some common strategies instructors use for grouping students for classroom work:

- Balance student personalities so that more vocal students are grouped with less vocal students.
- Regroup students often so that they work with a variety of students from class.
- Use different grouping strategies, such as using random generators, drawing from a deck of cards (all aces together, etc.), assigning groups based on the order in which students entered class, etc.
- Use more strategic approaches, such as grouping based on declared majors or interests, class performance, or other knowledge of the students.
- Avoid allowing students to remain in groups when the dynamic of the group impedes student learning. This is a good time to regroup!

Effective small group learning should incorporate tasks or questions that involve the five critical elements mentioned in section CP.1.5: positive interdependence, face-to-face interactions, individual accountability, social skills, and group processing.

In summary, collaborative learning entails students working on tasks in small groups of 3-4 students. In a more structured cooperative setting, students may be assigned roles in their groups such as facilitator, summarizer, recorder, and presenter. During this work, the instructor typically listens to the discussions and engages with different groups in a variety of ways, such as asking guiding questions or redirecting students' inquiries. Often a period of group work is followed by group presentations to the entire class, and the instructor may use the presented work to draw out important topics, to make connections, or to lead into subsequent material. The following vignette illustrates these processes.

Classroom vignette: Group work with rings

In his abstract algebra course, Professor Morales wants to help students generalize ideas that have been familiar to them since grade school. For example, a ring is a set of objects together with two operations, addition and multiplication, such that a certain collection of rules is satisfied. To illustrate the general behavior of elements in a ring, one can make analogies to the behavior of the integers under addition and multiplication. But how does one recognize when a student truly understands definitions such as "unity element" and "additive inverse"?

As is customary, the textbook uses the generic notation "0" and "1" to denote a ring's additive identity and unity, and the notation "$-a$" to denote the additive inverse of the ring element . The first surprise comes when students work on the following task in small groups:

Task 1. Calculate 0, 1, and -1 in the ring $\mathbb{Z}_4$, the set $\{0, 1, 2, 3\}$ under addition and multiplication modulo 4.

Misconceptions are evident. Students correctly identity the additive identity as the integer 0 and the unity as the integer 1, but many students insist that there is no additive inverse of the unity element since the integer -1 is not present in the set $\{0, 1, 2, 3\}$. This brings up an opportunity for a whole-class discussion of what the symbol "-1" means in an abstract setting. Professor Morales writes the equation $x+1=0$ on the board and asks the students to discuss in their groups whether this equation has any solutions in $\mathbb{Z}_4$. In this context, most students are comfortable with the fact that $x=3$ is a solution, and so writing $-1=3$ makes sense in this particular ring. Groups who see this right away are encouraged to generalize to the additive inverse of other elements in any ring $\mathbb{Z}_n$.

Professor Morales presents the next task for group discussion:

Task 2. Calculate 0, 1, and -1 in the ring $M_2(\mathbb{R})$, the set of 2×2 matrices over the real numbers.

Many groups will be wrestling with the use of symbols 0, 1, and -1 to represent matrices. For groups stuck on the values of 0 and 1, Professor Morales points out that a unity element satisfies

(1) In $\mathbb{Z}$, $a1=1a=a$ for all $a\in\mathbb{Z}$, and

(2) In $M_2(\mathbb{R})$, $A1=1A=A$ for all $A\in M_2(\mathbb{R})$.

Reminding students that in (2) the ring multiplication may only be between 2×2 matrices leads to the correct characterization of I as the unity 1 in $M_2(\mathbb{R})$, and then the correct interpretation of -1 follows the same way as in the previous task.

After building this facility to work within , Professor Morales next introduces a new task after a brief presentation to explain that the notation represents the repeated addition of copies of the ring element , provided is a positive integer.

Task 3. In the ring $M_2(\mathbb{R})$, compute the element $3\cdot 1$.

Professor Morales asks students to explain their reasoning very carefully within their groups. Most groups produce a correct answer very quickly, but upon listening to their explanations within their groups, the professor discovers that most of the students adopted scalar multiplication rather than repeated addition. Rather than lecture on improving the explanation, Professor Morales presents a final task for group discussion that renders impossible the incorrect explanation.

Task 4. Let $R=\{v, w, x, y, z\}$ be a ring where $v, w, x, y,$ and z are all distinct elements under the operations $+$ and $-$ for which the Cayley tables are given in Figure 4.

$+$	v	w	x	y	z
v	w	x	z	v	y
w	x	z	y	w	v
x	z	y	v	x	w
y	v	w	x	y	z
z	y	v	w	z	x

$*$	v	w	x	y	z
v	w	z	v	y	x
w	z	x	w	y	v
x	v	w	x	y	z
y	y	y	y	y	y
z	x	v	z	y	w

Figure 4. Cayley tables for Task 4.

Use the given information to complete the following.

1. Calculate 0 and 1 in this ring.
2. Calculate -1 in this ring.
3. Calculate $3\cdot 1$ in this ring.

Within this context there is no way to obtain a correct answer for without interpreting the expression as repeated addition. Moreover, one is forced to recognize the critical properties of the additive identity and the unity element in order to correctly identify them among all the ring elements.

After completing Task 4, students are asked to revisit their explanations for their answers to Task 3.

Discussion. In this vignette, we see how an instructor uses an activity to learn about his students' understanding of a specific mathematical concept. We will describe some of the aspects of a typical group work activity by considering the role of the instructor and the role of the student in this example.

An important aspect of the work of the instructor in group work activities is to select a task for students to work on.

- To elicit positive interdependence, select a sufficiently subtle or complex accessible task that generates discussion. Since the instructor cannot talk to all groups simultaneously, students need to have an entry point to productively engage with the material. In this vignette, the problem starts with a familiar example and then increases in abstraction.
- To attend to the mathematics, select an activity with a clear mathematical purpose in mind. In this case students generalize the idea of additive and multiplicative identity from the familiar ring of integers to more abstract rings.
- To expose student thinking, select a task that provides opportunities for formative assessment. Instructors can learn about his students and make informed decisions by listening to their students and by asking for explanations that may consist of flawed reasoning even if reaching a correct answer. Such inquiry can lead to task extensions that require a different strategy, such as the addition of Task 4 in the vignette.

During the activity, the instructor actively monitors and observes the group work in progress.

- Based on the student reasoning made visible during the group work, the instructor can decide where to go next. By observing his students work in groups on each task, he gains valuable insights into where his students have conceptual difficulties.
- The instructor selects approaches and ideas to highlight during whole class discussions and determines a sequence in which the approaches will be mentioned. Note that some parts of the activity may not need any attention—students sufficiently covered the content already—and other parts may offer opportunities for deeper learning or show gaps in students' understanding.
- During a whole class discussion, the instructor can make connections between ideas that have come up. In this example the instructor points out the parallels between how 1 (integer unity) and I (matrix unity) are solutions to similar equations, generalizing the idea of unity from specific examples to the general definition.

Managing the student engagement and interaction is also an important role for the instructor.

- Students work on problems in small groups, discuss solutions with each other and with the instructor. They encounter hurdles, questions, misconceptions. All students actively engage with each other and with the content—there is no place to hide and just watch. In the vignette, many groups have trouble with the notation used for the unities when applied to matrices. This is addressed directly within the group discussion before it becomes a major stumbling block.
- Students engage with the instructor by asking questions or giving explanations for their work. In the vignette, different groups justify their answers to the instructor in similar, incorrect ways. This points out widely held misconceptions that can be addressed during the activity and with a subsequent new activity.
- Students share solutions and often different groups contribute different approaches, contributing to

deeper learning for all students.

- Making connections to classroom norms and practices for mathematical problem solving and communication can enhance the equity-oriented considerations for the small-group work classroom practice. For example, how are members of small groups engaged in collaborative learning in ways that foster equitable, meaningful engagement with the mathematics?

CP.1.6. Just-in-time teaching (JiTT)

Just-in-time teaching (JiTT) is a formative assessment practice employing pre-class readings and online quizzes to shorten the feedback loop between out-of-class and in-class experiences. It was developed by Gregor Novak and Andrew Gavrin, physics professors at Indiana University-Purdue University Indianapolis (IUPUI), and Evelyn Patterson, physics professor at the Air Force Academy, in 1996. Their objective was to help students structure their out-of-class study time in order to get more out of the limited face-to-face class time. To accomplish this objective, they designed an online system that allows them to ask students questions and receive student responses in a short time frame. They used student responses to modify their classroom instruction "just in time," i.e., right before class started.

JiTT creates a short feedback loop between out-of-class and in-class experiences. Students prepare for class by answering questions on pre-class assignments, instructors modify their plans for in-class activities to address gaps in understanding and capitalize on student strengths, and instructors reflect on in-class activities to develop the next set of pre-class questions. Additionally, instructors can provide individual students with feedback on their responses. Specifically, many JiTT platforms and learning management systems allow instructors to send feedback to students directly from the response-viewing screen.

The questions on pre-class assignments, often referred to as "JiTTs," should be "short, thought-provoking questions that, when fully discussed, often have complex answers" (Novak and Patterson, 2010, p. 6). They should have a low floor and a high ceiling. That is, they should be simple enough to allow students to engage with them despite not yet having formal instruction on the concept, but complex enough to spark interest and require meaningful thought to answer. The *GoodQuestions Project*, run by researchers at Cornell University, developed many multiple-choice questions for use as JiTTs in calculus. For example, the question, "Were you ever π feet tall?" could be posed in a calculus class to make students grapple with continuity, the intermediate value theorem, and irrational and transcendental numbers. To help students develop their metacognitive skills, instructors often end their JiTT assignments with a question such as, "After completing this exercise what concepts or ideas are still unclear and why?" (Simkins and Maier, 2010, p. xvii).

JiTT differs from online homework because it is pre-class formative assessment, designed to get students thinking about an idea before class. On the other hand, homework (whether online or paper-based) is post-class assessment, usually summative, designed to help students cement and demonstrate their understanding of concepts discussed earlier in class. Given that JiTT is a formative assessment practice and JiTT exercises are "non-judgmental diagnostic tools" (Novak and Patterson, 2010, p. 11), student responses to JiTTs should be graded solely on the basis of effort and completion, rather than on the basis of correctness. This allows students to feel safe expressing ideas that are still tentative, inchoate, or under development. As illustrated in the second vignette below, incorrect or partially correct responses are often extremely valuable as seeds for classroom discussion. Additionally, incorrect responses can help illuminate gaps in students' understanding of a particular topic. That is, if a large number of students all make the same mistake, then instructors know to focus their in-class activities in an effort to address that mistake.

Research (see Formica et al., 2010; Marrs and Novak, 2003; Riskowski, 2014) suggests that students in JiTT classes exhibit increased preparation for and participation in the in-class activities due to having spent some time before class thinking about the material. They feel more ownership of important ideas and concepts, because their own words are used to start the in-class conversation, and they have frequent opportunities to practice discipline-appropriate reasoning, communicating, and metacognition in a low-stakes environment. Instructors develop a more complete picture of what topics students do and do not fully grasp, are able to modify their instruction to target more directly the concepts students are having the most difficulty with, and have another opportunity to give students quick feedback on their learning. Furthermore, empirical results show that students in JiTT classes outperform their peers in similar classes without JiTT .

We illustrate JiTT with two vignettes and follow-up discussions, and then provide more information on implementing JiTT in the classroom.

Classroom vignette: An all-too-familiar story

Dr. Gomez taught her calculus class the formal definition of continuity (i.e., that f is continuous at $x = c$ if $\lim_{x\to c} f(x)=f(c)$) and discussed the use of this definition to compute limits: if we know that f is continuous at some point c (for example, if f is a rational function for which the denominator is nonzero at c), then we can compute $\lim_{x\to c} f(x)$ by simply evaluating the function. However, she continued, if we are going to compute limits in this way, then we must justify our steps by saying why we know f is continuous at c.

Her students appeared to be engaged during her lecture, nodded along through several examples, and even worked together in small groups through an example problem of their own. Dr. Gomez, consequently, believed that her class understood this idea. However, as Dr. Gomez graded the test she gave her students a week after this class period, she quickly discovered that very few people justified their steps in limit calculations. She was both surprised and dismayed by this finding, because she thought students understood the idea so well during the class session and because the students had high scores on the online homework. As a result, she spent a fair amount of class time discussing this common mistake even though she felt that her students were confused or disengaged because it had been so long since their original exposure to the formal limit definition of continuity.

Discussion. The problem that manifests itself in the above vignette is that the feedback cycle took too long. Students learned the limit definition of continuity, a week elapsed before the test, several days of grading time elapsed further, and Dr. Gomez was thus unable to identify and address the misunderstanding until perhaps two weeks after students' initial introduction to the idea. By that time, her students had learned so many other things that the formal definition of continuity was no longer fresh in their minds, and thus they were not receptive to the re-discussion of the idea.

Classroom vignette: Another story, with a shorter feedback cycle

On Friday, Dr. Gomez planned to cover global optimization of continuous functions. On the preceding Wednesday, she sent her students a link to a pre-class assignment due one hour before class on Friday, in which she asked students to look at the following graph of a function $f(x)$ shown in Figure 5.

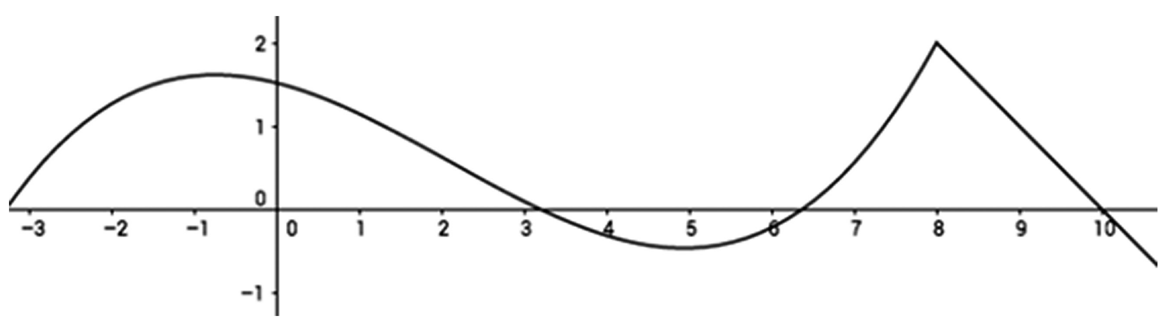


Figure 5. Graph of $f(x)$

On this pre-class assignment, Dr. Gomez asked students to estimate the critical points and critical values of $f(x)$, and to identify the very smallest and very largest values of $f(x)$. The students were also asked to identify the very smallest and very largest values of f if its domain was restricted to $[-3, 5]$ and if its domain was restricted to $[0, 4]$. Finally, the students were asked to make a conjecture based on their work about the possible locations of global extrema of a function defined on a closed interval.

In class on Friday, Dr. Gomez highlighted two student conjectures: Student A's conjecture was, "Global extrema occur at critical points," and Student B's conjecture was, "The global extrema are the endpoint values." She asked her students to decide whether Student A or Student B was correct. After exploring several examples in small groups, her students decided that both Student A and Student B were partially correct. Dr. Gomez next asked, "How could we change their statements to make them *completely* correct?" and she gave the students several minutes to discuss this question in small groups. Eventually, the class decided that Student A should have said, "Global extrema *can* occur at critical points," and that Student B should have said, "The global extrema *can be* the endpoint values."

Dr. Gomez continued with the rest of her plan for the class session, which included students working together in small groups to find the global extrema of $f(x) = x^3 - x^2 - 48x + 52$ on the domain $[0, 10]$ and on the domain $[-5, 15]$. In both instances, students automatically checked the values at the endpoints, saying to each other that they had to do that because of Student B's pre-class assignment response.

Discussion. Compare the length of the feedback cycle in this example to the length of the feedback cycle in the first example. In the second story, the assignment was only out to students for a period of several days; grading the assignment only took a few minutes, since it was on the basis of completion and effort; and students received feedback in an anonymous and holistic way during the very next class session, when the concept was still fresh in their minds. The second story is an example of Just in Time Teaching, JiTT.

Practical tips

As with any of the other student engagement strategies detailed in this guide, JiTT can be implemented well and have good results, or it can be implemented poorly and have poor results. In particular, if pre-class activities and questions are not carefully integrated into the in-class activities, students may come to resent the extra work, which can lead to negative effects on their motivation and attitudes toward the class, and even to the possibility of a reversal of learning gains (Camp et al., 2010).

When implementing JiTT, the following are needed: (1) JiTT questions that are thought-provoking for students, (2) a platform where students can respond to JiTTs and where the instructor can grade their responses, and (3) a plan to incorporate JiTT responses into in-class activities.

- **Question banks:** The *GoodQuestions Project* (www.math.cornell.edu/~GoodQuestions/) is a source of JiTT questions for calculus, and the *JiTT Digital Library* (jittdl.physics.iupui.edu/) provided by the original inventors of JiTT focuses primarliy on physics concepts. Additionally, *ConcepTest* questions designed for use in peer instruction can often be modified to be good JiTT questions, and many textbook publishers provide extensive banks of *ConcepTest* questions carefully aligned with the textbook. Finally, the website mathquest.carroll.edu/resources.html contains links to resource collections primarily focused on classroom voting systems.
- **Platforms:** Most learning management systems (such as *Canvas*, *Blackboard*, *D2L*, or *Moodle*) and online homework systems have integrated assignment features that are excellent for collecting and grading student responses to JiTT questions. Additionally, many instructors use online survey tools (such as *Google Forms*, *Qualtrics*, or *SurveyMonkey*) to accept and grade student responses. Of particular interest is a tool purpose-built by the JiTT developers; a guide to this tool is available at bit.ly/jittdlguide.
- **Planning:** There are two places where instructors will need to do some careful planning. The first is the beginning-of-semester dialogue conveying class expectations. Because JiTT is a formative assessment technique and, thus, quite different from the summative assessments most students are used to,

instructors will need to spend a fair amount of time explaining it at the beginning of the semester. Instructors should plan to explain to students the benefits of the system, give them tips on how to make sure they are keeping up with the daily JiTT activities, and clarify expectations for their answers. Second, instructors will need to plan how to integrate JiTT responses into daily class activities. Instructors often schedule five or ten minutes of JiTT response discussion into the beginning of each class session and look for points in each class session that can be illustrated nicely by a student comment. Avoid positioning individual students as being more or less mathematically able when orchestrating whole-class discussions based upon JiTT responses (see Ray, 2013, pp. 42–55).

- **A caveat:** As with any of the student-centered strategies outlined in this guide, it is best to start small. Instructors who attempt to make a pre-class assignment for *every* class the first time they try JiTT may become overwhelmed with this technique. It is usually more successful to develop and improve the list of pre-class assignments iteratively each term by perhaps integrating a JiTT prompt once every two weeks and adding and refining questions each term until the set of pre-class assignments becomes manageable and stable.

CP.1.7. Developing persistence in problem solving

A possible obstacle to student-centered classrooms relates to students' learned or innate ideas about mathematics and what it means to do mathematics. Many tasks meant to actively engage students in the classroom work best if students understand that persistence in working mathematics is valued and is integral to doing mathematics. **Persistence** can be defined as "student actions that include students concentrating, applying themselves, believing they can succeed, and making effort to learn" (Clarke et al., 2014, p. 67). The good news is that according to research, perseverance can be improved (Duckworth, 2016).

Classroom vignette: Persistence

Dr. Smith has been working with his analysis students on the basics of mathematical sequences. His goal is for students to understand the intricacies of the formal definition for the limit of a sequence: "A sequence $\{a_n\}$ has a limit L if for every $\varepsilon > 0$, there exists a natural number N such that for every $n > N$, $|a_n - L| < \varepsilon$." To this end, he wants his students to re-invent this definition on their own, a task that is quite challenging and requires a good deal of persistence. As class begins he enthusiastically explains the task and acknowledges that it will require persistence. "After all," he says, "inventing the formal limit definition took mathematicians over 100 years!" However, he also tells the students he believes in their ability to succeed and promises to guide them away from any dead ends so that the project will take only a few weeks rather a century. He builds their motivation by explaining why the mathematics is useful and by connecting it to their personal interests and goals (Schechtman et al., 2013). Since this is an analysis class for mathematics majors, he takes a different motivational route than he does in his college algebra class. He explains that they will be working like mathematicians, and that this task will greatly enhance their ability to understand and to do upper-level mathematics.

Dr. Smith has learned that how he sets up the task is crucial and that an overview of how the task will play out can make a significant difference in helping students persist. As such he begins by asking students to create a set of sequence graphs, some of which converge to 5 and some of which do not. He explains that students will write an initial draft definition that describes the sequences that converge to 5 and "kicks out" the sequences that do not. The students test their definition against their sequence graphs and refine their definitions to take care of any problems they notice. At this point, the cyclic process of "testing and refining" will go through many iterations until the students develop a complete definition.

Dr. Smith has the students work in groups of four to provide the peer support needed to persist. As the groups begin to work he wanders around the room, observing and listening carefully. He listens for areas of difficulty while resisting the temptation to tell them what to do so that students will feel their definitions are their own. He notices that most groups create definitions such as, "A sequence converges to a limit of 5 if it *approaches* 5." After about 10 minutes he brings the class back together to briefly discuss the vagueness of the word "approaches" and the need to avoid circular definitions. He makes sure the discussion is brief because he knows that giving the students plenty of time to work on the task will encourage persistence. As the students continue working, Dr. Smith makes sure to talk, stand, and look in ways that communicate that it matters to him what each student is saying and doing even if he does not comment (Lampert, 2001). After a few minutes, he notices that Brian, Jeff, Kayla, and Misty have developed the following definition, "A sequence converges to a limit of 5 if the next term always moves closer to 5." Resisting the urge to immediately point out the graph of the damped oscillating sequence or the fact that a sequence like $a_n = 4 - 1/n$ satisfies their definition, he instead asks a guiding question, "Does your definition work for all of our sequence graphs that converge to 5?" or "Does your definition also work for sequence graphs that do not actually converge to 5?" After testing the definition against each graph, the group realizes *on their own* that the damped oscillating sequence is a problem or that the sequence $a_n = 4 - 1/n$ satisfies their definition, but converges to 4.

Although watching the students struggle can be difficult and time-consuming, Dr. Smith knows that giving challenging tasks at the right level can facilitate persistence. After a while, he notices that one group struggles more than the others and one outspoken student from this group is extremely frustrated. He talks to this group about having a growth mindset (Dweck, 2008; also see the equity section in the chapter on Cross-Cutting Themes), explaining that while this is a difficult task, they can actually grow their intelligence by persisting even if they do not make as much progress as they would like. He modifies their task for now, suggesting that they focus on writing a definition only for the monotonic sequences and then include the more difficult sequences once the monotonic ones are tackled. Towards the end of the class period, each group shares their initial definitions, discusses their ideas, and describes problems they encountered. The other class members have a chance to ask questions and give feedback. This gives everyone a chance to glean ideas from the other groups' definitions and to celebrate their persistence.

Practical tips

A number of strategies exist for motivating students to attempt complex tasks and to persist until they solve them.

- Choose challenging tasks that are beyond students' current abilities because they can motivate students (Seeley, 2009). These tasks should require students to struggle constructively. In contrast to frustration, constructive struggle involves expending effort to make sense of the mathematics and to figure out something that is not readily apparent (Heibert and Grouws, 2007).
- Set up the lesson carefully (Clarke et al., 2014). Be enthusiastic about the tasks. Acknowledge that the tasks require persistence. Give students a brief overview of how the lesson will play out. Make your expectations clear. Believe in each student's ability to succeed. According to the Carnegie Foundation for the Advancement of Teaching, a student's belief that they are capable of learning mathematics is an important psychological driver of persistence (see the equity section of the chapter on Cross-Cutting Themes).
- Provide opportunities for intrinsic rewards and appropriate extrinsic rewards. Extrinsic motivators such as grades can damage persistence, but intrinsic motivation based on individual interests and a desire to grow and learn can strengthen perseverance (Clarke et al., 2014). Build intrinsic motivation by focusing on the value of the task. Connect the task to students' everyday lives, interests, and goals (Schechtman et al., 2013). Use extrinsic rewards only if these rewards are unexpected and encourage identifiable behaviors rather than outcomes (McKay, 2015).

- Encourage students to develop and value a growth mindset instead of a fixed mindset (Dweck, 2008). Specific to mathematics, a fixed mindset implies that students believe that their talent in mathematics is fixed. That is, one is either good at mathematics or one is not. A growth mindset in mathematics implies that students view challenge and struggle as avenues to shape and expand their mathematical understanding and that mathematical talent is malleable.
- Give students plenty of time (Clarke et al., 2014). To encourage persistence, the lesson must be structured so that students have plenty of time to struggle. Some educators have suggested giving students a few minutes to plan with a partner how they will solve the problem. Keep any mini-lessons brief and to the point.
- Have students work in groups. Feeling socially tied to peers is another important psychological factor that drives persistence (Carnegie Foundation for the Advancement of Teaching). Actively monitor status issues related to gender or race that may impede full participation in the group or possibly cause a situation where Stereotype Threat impedes developing persistence in a social setting.
- Resist the urge to tell (Clarke et al., 2014) and allow students to struggle. Lampert (2001) suggests using about two-thirds of the problem-solving time to move around the room, simply watching and listening. Look and stand in ways that show that it matters to you what they are saying and doing even if you do not comment. Instructors may need to develop questions on the spot that guide students but "stop short of telling [them] what they need to know to solve the problem" (Seeley, 2009, p. 90). Any teaching that is done will be "on the run" in response to what you see or hear (Lampert, 2001). Think about how to support student engagement without removing the struggle (Baldinger and Louie, 2014). As you listen, push students to justify their thinking and strongly encourage them to write down their justifications (Clarke et al. 2014). As you walk about the room, select certain students to present their work during the closing stage of the lesson and plan the sequence in which their work should be displayed.
- As needed, pull the class back together for brief discussions or mini-lessons (Clarke et al., 2014). A good time to bring the class back together for a short discussion might be after about 10 minutes of problem-solving. If students have been given a series of problems to solve, ask for their solutions to the first few, having the *students* collectively decide on the correct answers. If students are tackling just one problem during the class period, ask them to describe and discuss their initial thoughts and attempts. If you observe common difficulties or notice that some students are struggling more than others, bring the class together for a short mini-lesson. If the lesson is brief, the students who have demonstrated a strong mathematical understanding and successfully completed the problem solving task will be happy to see they are on the right track.
- Prepare appropriate prompts ahead of time to support persistence for students at various stages in their mathematical understanding. Prepare "enabling prompts" to help students who are struggling with the entry point into the mathematical task (Sullivan, 2011). These can be slightly different initial tasks that can get students on track before returning to the main problem. Prepare "extending prompts" for students who complete the task quickly to keep them appropriately challenged, and pace group interactions (Sullivan, 2011). After the lesson reflect on how future instruction could create opportunities for more students to engage deeply with the mathematics (Baldinger and Louie, 2014).
- As appropriate, talk to the students about the metacognitive strategies involved in mathematical problem-solving. In order to persist, students need to be able to step back and monitor their progress. Help them develop strategies for getting "unstuck." When feeling frustrated they may want to stop and take a look at their progress, consider if they are on the right track, determine if they need to switch strategies, or perhaps get some pointers from a peer or from the instructor.
- As appropriate, close the lesson with a summarizing discussion (Clarke et al., 2014). Have selected

students present their solutions. Help the class make connections between different students' responses and summarize their work. Keep in mind that summarizing discussions could occur at various points during the lesson or could occur at the beginning of the next class period. Remember that even incomplete solutions or incorrect responses may reveal important points. Remind the students of the important aspects of a growth mindset, such as the fact that errors are part of learning, and learning occurs even if the task is not completed.

- Celebrate students' learning and be proud of their efforts to persist!

CP.1.8. Inquiry-based teaching and learning strategies

Inquiry-based teaching provides a rich way in which to actively engage students in the classroom. In the spirit of inclusiveness of a range of particular teaching strategies, we offer three guiding principles that promote the success of inquiry-based teaching: students' deep engagement in mathematics, peer to peer interaction, and instructor interest in and use of student thinking (Hayward and Laursen, 2014; Rasmussen and Kwon, 2007). Specific strategies for enacting these principles are within the purview of the individual instructor. Below, we elaborate on each of these three guiding principles.

Deep engagement in mathematics

In an inquiry-based learning classroom, students are engaged in doing mathematics. They are not just listening and taking in information. Rather they are engaged in deep mathematical thinking. This does not mean that lectures do not exist in every IBL classroom, for there is a time and place for well-crafted exposition that brings ideas together and makes connections to more formal or conventional mathematics. Lecture, however, does not mean doing problem after problem for students. Instead, the instructor may introduce a new topic and have the students try problems themselves. The students try unfamiliar or familiar problems, but are engaged in productive struggle until they figure out the answers themselves or with guidance from the professor or other students. We also refer readers to the section on Selecting Appropriate Mathematical Tasks (see section CP.2).

Peer to peer interaction

Given that students are working on problems that are designed to be engaging, it often means that these problems are also more difficult than standard problems and require collaborating with peers. Working with others can appear in different forms (in or outside of class) such as group work, think-pair-share, collaborative board work, whole class discussion, etc. In all of these settings students communicate mathematics with each other via mathematical discussions which can include arguing about the mathematics. These collaborations facilitate learning to form logical arguments and as a result students are able to tackle more difficult problems.

Instructor's interest in and use of student reasoning

In a classroom where an instructor adopts inquiry-based teaching, students are deeply engaged in doing mathematics and collaborating with their peers. Such work on the part of the students provides the instructor with ample occasions to inquire into students' reasoning, to help students listen to and orient to other students' reasoning, and to use students' reasoning to advance the mathematics lesson. Such work on the part of the instructor necessitates interest in how students reason, and it creates a classroom environment where students explain their own reasoning and attend to the reasoning of others. Fortunately, the research literature provides some helpful and concrete suggestions so that instructors can achieve these goals (e.g., Michaels and O'Connor, 2013; Rasmussen, Yackel, and King, 2003, pp. 143–154). For example, if a student participates in the discussion, the student has to be able to share thoughts and responses out loud, regardless of how tentative or unsure they are of their ideas. If only one or two students can do this, it is not a discussion—it is a monologue or, at best, a dialogue between the instructor and a student. Prompts that facilitate a classroom environment where students routinely share their reasoning include:

- "Say more about that."
- "Dave, I know you haven't finished, but tell us your initial thinking."
- "Take your time, we're not in a rush."
- "Oh, that's interesting. Did everyone hear what Julie just said? She said that …"
- "That's an important point. Keisha, can you say that again?"

Once students start sharing their reasoning, they need to consider others' ideas and reasoning and respond to them. This is the impetus of a real discussion that supports robust learning. Given students are used to instructors evaluating and commenting on their reasoning, it takes explicit attention on the part of the instructor to help students attend to and make sense of other students' reasoning. Prompts that can help instructors achieve this goal include:

- "Who can repeat in their own words what Juan just said?"
- "But what about what Damen's point that …?"
- "Oscar, can you say in your own words what I am asking you all to do?"
- "So Debbie, is Jorge saying that …?"
- "Jesika, what do you think about what Ding just said?"

Note that some of these prompts are ones that a person typically does not use in everyday conversation but are extremely useful for instructor talk.

Inquiry-Based Learning (IBL) is used as a student-centered teaching strategy throughout all levels of instruction in mathematics and in other STEM fields. IBL engages students in sense-making activities. Students work on tasks that require them to solve problems, conjecture, experiment, explore, create, and communicate, all critical skills and habits of mind in which mathematicians and scientists engage regularly. Rather than showing facts or a clear, smooth path to a solution, the instructor solicits student reasoning and guides students via well-crafted problems and questions through an adventure in mathematical discovery while using student reasoning to advance the mathematical agenda. Students are thus engaged in the creation of mathematics, allowing them to see mathematics as a part of human activity, not apart from it (Ernst, Hodge, and Yoshinobu, 2017; Freudenthal, 1991).

CP.1.9. Peer instruction and technology

The term **peer instruction**, coined by Harvard physics professor Eric Mazur, refers to a particular sequence of in-class activities modeled on the think-pair-share sequence and supported by the use of a classroom response system. Some of these systems use handheld devices (referred to as "clickers") that let students submit their answers to a receiver plugged into the instructor's computer. One resource that can be leveraged for more information about clickers can be found at cft.vanderbilt.edu/guides-sub-pages/clickers/. Other systems use a "bring your own device" approach that requires students to respond via wifi or text messaging using their own devices such as phones, tablets, or laptops. Instructors can even simply supply colored or numbered cards for students to hold up in order to accomplish a no-tech version of this idea.

Peer instruction typically involves the following steps:

1. An instructor poses a question, often a multiple-choice question, aimed at differentiating student con-

ceptions about a topic.

2. All students in the class are invited to answer the question using the classroom response system.
3. If there is a technological system involved, the instructor views the aggregated responses of the students, perhaps as a bar graph generated by the system.
4. Based on the student responses, one of three things happens next.
   a) If most students answer the question correctly, then the instructor can move on fairly quickly to the next topic. Some brief discussion of the question is warranted for the sake of students who answered incorrectly or guessed their way to the correct answer, but if 80% or more of the students got the question correct, it may be a better use of class time to move onto other topics.
   b) If students are split among two or more answers, then the instructor asks students to turn to their neighbors, discuss the question, and try to reason their way to the correct answer. After some time for this paired discussion, the instructor asks the students to answer the question a second time using the classroom response system, perhaps sticking with their original answer, perhaps changing their response based on their paired discussion. After this, the instructor displays the aggregated responses to the students and leads a classwide discussion of the question, inviting students to share their reasons for the answers they chose and guiding the conversation toward the correct answer—and the correct reasoning.
   c) If most students answer the question incorrectly, having students pair up and discuss the question is not likely to be useful. In this case, the instructor may ask a question to probe student understanding and provide clarification or pose some scaffolding questions or examples to help students bridge ideas and reach a consensus.

Most instructors who practice peer instruction aim for that middle outcome by crafting questions designed to be hard, but not too hard. It is in this "Goldilocks zone" where peer instruction shines. "Just right" questions that are not too hard and not too easy.

Classroom vignette: Peer instruction and classroom response systems

At the beginning of a unit on probability in an undergraduate statistics course, Dr. Sun poses the following question to her students. *Your sister calls to say she's having twins. Which of the following is more likely?* (Assume she's not having identical twins.)

A. Twin boys
B. Twin girls
C. One boy and one girl
D. All three are equally likely.

Dr. Sun asks her students to get out their phones, to log on to the classroom response system, and to respond individually to this question. "No talking right now," she says, encouraging the students to think for themselves and commit to an answer. After about 30 seconds, the response system says that 27 of the 30 students in the class have submitted their answers, so Dr. Sun says, "Last call!" The remaining three students respond quickly, and she looks over the bar chart generated by the system. Fourteen of the students (47%) have answered D ("All three are equally likely), while 12 of the students (40%) have answered C ("One boy and one girl"), the correct answer. The remaining students are split between choices A and B.

She then displays the bar chart to the students on the classroom projector. "There is not a consensus here," she says and asks the students to pair up and discuss the question. "Talk about the reasoning you used in arriving at your answer. Even if you both agree on the answer, you may have different approaches that provide insight into the problem." The room buzzes as students start arguing about the answer, some sharing their personal experiences with twins, some drawing genetic diagrams they saw in their biology courses, others arguing over whether boy babies are more common than girl babies. After a couple of minutes, Dr. Sun asks students to answer the question again using their phones. "You can stick with your original choice," she says, "or you can change your choice based on your discussion." The student responses come in quickly, and she displays an updated bar graph. This time, 22 of the students (73%) have the right answer: C, one boy and one girl. Only seven students (23%) still answered D. One student stuck with "twin boys," in spite of the popularity of answers C and D.

Given the results, Dr. Sun asks for a volunteer who changed their mind from D to C to explain their reasoning with the class. In an effort to encourage more students to engage she calls on a student who is usually quiet in class. That student explains that she thought that all three outcomes were equally likely, but her partner convinced her that there were actually *four* equally likely outcomes: boy-boy, boy-girl, girl-boy, and girl-girl. And that meant that there was a 50% chance of one boy and one girl. Dr. Sun knows this explanation is a correct one, but she does not want to confirm the right answer for the students too soon. She wants to make sure they all have a chance to reason it out themselves. Thus, she asks for another volunteer, this time someone willing to explain their reasoning for "all three are equally likely." Another student speaks up, saying that having a boy or a girl is 50/50, so all three combos are equally likely. "If we had identical twins in the mix, then a boy-boy or girl-girl combo would be more likely," the student says, "but we don't, so the right answer is D."

At this point, Dr. Sun is confident that several students are still unsure, so she decides to run a simulation. She asks each student to get out a coin (or a student ID card), assign one side to "boy" and one side to "girl," then flip the coin (or ID card) twice, to simulate having twins. Again the classroom starts buzzing, as students flip coins and other items at their seats. Then she asks the students to report their results using the classroom response system: A for twin boys, B for twin girls, C for one of each. This time, she lets the bar chart update itself on the big screen in real time as the data come in. After a few seconds of moving bars, the chart settles down on the final distribution: 23% twin boys, 30% twin girls, and 47% one of each. "Given this admittedly small sample size," she says, "would you believe that not all three are equally likely?"

To finish the discussion Dr. Sun draws a quick tree diagram on the chalkboard, noting the two simplifying assumptions she is making with this question, that there are only two sexes (boy and girl) and that they occur with equal probability. Twin #1 is either a boy or a girl, with equal odds, and Twin #2 is either a boy or a girl, again with equal odds. The tree diagram illustrates the four equally likely outcomes mentioned by the first student volunteer, and Dr. Sun confirms that boy-girl will occur 50% of the time, on average.

"Why did I have you spend 20 minutes of class time working through this example?" she asks, rhetorically. "Because a lot of people have the misconception that with probability, all outcomes are equally likely. That's not always the case, especially when we group multiple outcomes into what we call events, as we did here. We needed to directly confront this misconception, if you're going to make sense of this probability unit we've begun. We humans tends to have a lot of misconceptions about probability, so we'll have to model the problems in this unit very carefully as we go."

Discussion. There are a number of reasons discussed elsewhere in this guide that indicate why collaborative learning activities, such as peer instruction, are effective at promoting student learning. Below we offer reasons as to why and how technology such as classroom response systems can support peer instruction.

Formative Assessment. In contrast to the summative assessment performed at the end of a course for evaluative purposes, formative assessment involves making student learning visible during the learning process, so that students can receive feedback on their learning. A classroom response system is a tool for formative assessment that does not depend on the class size. All students are invited to respond to questions and to commit to their answers, and thus all students have the chance to find out if their answers are correct. This type of feedback is critical to the learning process, and a classroom response system helps it occur regularly during class.

Agile Teaching. Formative assessment is not only useful to the student, it is also useful to the instructor. Bar chart summaries of student responses allow instructors to determine if students have mastered a particular concept, whether they need more help with a topic, and even which misconceptions are most common in a particular class of students. All of this information, gathered "on the fly" during class allows an instructor to practice what is sometimes called *agile teaching*—making instructional choices in the moment that respond directly to student learning needs.

Times for Telling. Students are better able to understand and remember an explanation if they are ready to hear it. By asking all students to commit to their answers to a question and by showing students a distribution of answers that makes clear the question is a hard one, an instructor can use a classroom response system to create a "time for telling" (Schwartz and Bransford, 1998). Cognitively, students are ready to understand an explanation because they have already brought to mind their prior knowledge and experiences as well as conjectured about the question. Students want to know the answer because they have committed to their answers, and the bar graph shows them their peers disagree about the question. This can be a powerful process for surfacing and confronting student misconceptions.

All of this is possible without technology, but a classroom response system makes certain aspects of peer instruction easier, such as asking all students to respond individually before seeing their peers' responses. Furthermore, it allows students to discern whether there is agreement or disagreement in the response.

Practical Tips

The classroom vignette above featured a question designed to surface a misconception about a topic. Misconception questions work very well with peer instruction, but other types of questions are also possible, including application questions that require students to apply a computational algorithm to a given example or to choose the correct expression to set up a computation given a word problem. Also useful in some courses are ratio reasoning questions that ask students to determine if the value of a variable will increase, decrease, or stay the same if some other variable changes. See the *MathVote* website, mathquest.carroll.edu/, for question banks for commonly taught mathematics courses.

Some instructors follow the peer-instruction sequence, but skip the individual vote phase, preferring to have students move directly to the paired discussion. This can be useful for hard questions, where students are likely to need some discussion before they can formulate an answer, or when time is short. But it is usually better to have students answer individually before the peer instruction phase, so that every student has something to share during paired discussions.

When the vote is split among two or three answer choices, showing students the results can help motivate discussion, as the students see that the question is a hard one. However, if one answer is more popular than other answers (for instance, a 10/60/10/10/10 distribution), showing students such a bar graph can inhibit discussion, as students assume the popular answer is the correct one. In these cases, it usually is better not to show students the results, but peer discussion can still be useful, given how many students are unsure of the correct answer.

The examples above all assume the use of multiple-choice questions. Historically, "clickers" only allowed for the submission of answers to multiple-choice questions, so instructors practicing peer instruction were limited to such questions. In today's classrooms, BYOD (bring your own device) systems get around this limitation, making free-response questions more practical. However, aggregating results from free-response questions is still challenging. A simple bar graph works great for multiple-choice questions, but it does not work for short answer questions. Sometimes a word cloud can help instructors see patterns in student responses, but word clouds are not as useful for numeric response questions. Scrolling through a complete list of student responses is sometimes useful, especially in smaller classes, but in general, free-response questions are not as easy to use with peer instruction in a mathematics class as multiple-choice questions.

CP.2. Selecting appropriate mathematical tasks

Selecting appropriate mathematical tasks is critical for fostering student engagement. The tasks chosen provide the conduit for meaningful discussion and mathematical reasoning. But, how does an instructor know when a mathematical task is appropriate? There does not appear to be one single idea on what constitutes **appropriateness** in the research literature or in practice. Rather, appropriateness appears to be determined from a combination of a number of factors. The successful selection of an appropriate mathematical task seems to involve two related ideas.

1. The **intrinsic appropriateness** of the task, by which we mean the aspects of the task itself that lend itself to effective learning.
2. The **extrinsic appropriateness** of the task, by which we mean external factors involving the learning environment that affect how well students will learn from the task.

We now look at each of these ideas in turn.

CP.2.1. Intrinsic appropriateness: What makes a mathematical task appropriate?

By intrinsic appropriateness, we mean aspects that are inherent in the task itself that affect how well-suited it is for the moment. These include:

The degree to which the task is aligned with the learning objectives of the lesson and the course: A task that is in clear harmony with the learning objectives for the lesson and/or the course is more appropriate than one that is not. For example, if a lesson is designed so that a stated outcome of the lesson is the ability to find the roots of a second-degree polynomial equation, then a task requiring students to do this is an appropriate one; a task that requires students to find the roots of a linear or cubic equation is not. On a larger scale, if a high-level learning objective of the course is to use technology effectively to solve mathematical problems, then it would be appropriate to include technology as a means to solving a task. If the use of technology is *not* one of the course objectives, then having students perform a task with technology may not be appropriate.

The mathematical expertise of the learners: If an instructor selects a task for which students have not had adequate preparation, then clearly the task is not appropriate. For example, in a calculus class a task that involves the chain rule is not appropriate if the students have not yet learned the chain rule or if students have demonstrated issues with the concept of a composite function. Conversely, if a task is significantly below the expertise level of students, then it may not be appropriate because students may not be motivated enough to engage with it.

Student readiness coming into the task: This is a different idea than **mathematical expertise** although related. Whereas expertise refers to the knowledge that a student has on a subject, **readiness** refers to whether that knowledge is activated and ready for use. For example, a student might have expertise in solving second-degree polynomial equations using the quadratic formula, but may not be able to apply the quadratic formula in novel situations without guidance from an expert, such as the instructor. Instructors need to be knowledgeable about students' prior knowledge.

The degree to which the task satisfies students' basic cognitive needs and provokes intrinsic motivation: This is similar to the first point. A task that offers students no reason to be interested is likely not an appropriate choice. A more appropriate choice might be a task that has similar actions to perform (e.g., factoring a polynomial) but which is somehow connected to students' basic cognitive needs for **competence**, **autonomy**, and **relatedness**. Self-determination theory (discussed in section CP.2.3.) states that students' intrinsic motivation to complete a task requires a confluence of meeting these three needs.

The level and nature of cognitive load that the task places on the students: A task that is needlessly complicated will place a strain on students' cognitive load and could lead to a failure to engage with it. On the other hand, a task that addresses a learning objective with a minimum of **extraneous load**, or extra work, which adds **germane load** to the basic concept, will be better suited for student learning. We discuss cognitive load theory more in section CP.2.3.

CP.2.2. Extrinsic appropriateness

Whether a given task is appropriate for a given situation also depends on factors not inherent in the task itself, such as the following:

Student motivation coming into the task: A mathematical task may be at an appropriate level of difficulty and in alignment with instructional goals, and yet students may fail to engage with it in a meaningful way because of their levels of motivation. Apart from whether a task is inherently interesting or meaningful to the student, the deployment of a task will be more successful—more appropriate for the moment—if students recognize why the task is interesting.

The degree to which the physical space and makeup of the learning environment is suited to the task: For example, a task that involves students self-selecting into small groups of three of four may be entirely appropriate for a classroom in which the furniture can be rearranged easily, and yet not as appropriate for a class with fixed stadium-style seating. A task that involves students putting work on a chalkboard or whiteboard may not be appropriate for a class meeting in a space in which there are no such implements. A task that involves peer instruction, such as using classroom response devices to probe student misconceptions and provoke discussion, may be more appropriate for a large lecture course than for a tutorial with fewer than ten students.

The degree to which the mode of instructional delivery is suited to the task: Here we are mainly thinking of a distinction between face-to-face courses, online courses, and hybrid courses. While definitions of these terms vary, we informally refer to face-to-face courses as those in which the students in the class meet together at fixed times in a common, fixed location for all its main activities. Online courses are those in which the main class activities do not occur in a fixed space but rather in a common online locale such as a course management system. We can make a further distinction between synchronous online courses, in which students meet online but at fixed times, and asynchronous online courses in which there are neither fixed times nor fixed physical locations for course meetings. Hybrid courses are those that combine face-to-face and online components. Some tasks that might be appropriate for face-to-face courses would be difficult or impossible to accomplish in an online course. For example, exit tickets do not make sense for asynchronous online courses in which there are no meetings at a common time. Conversely, some tasks that leverage the online environment, such as having students work out solutions on a discussion board, might be less appropriate for a face-to-face course, where, for example, having students work out solutions in class might be more appropriate.

CP.2.3. Theoretical frameworks for understanding appropriateness

The above considerations are rooted in at least three major theoretical perspectives on human learning: Vygotsky's **Zone of Proximal Development** idea, **cognitive load theory**, and **self-determination theory**. Although there is much literature on each of these topics, we briefly describe each of these perspectives; the reader can find more in the Design Practices Chapter.

Zone of proximal development

Vygotsky (1978) developed the notion of the *zone of proximal development* (ZPD), which can be described roughly as the space between what a learner can do without help and what the learner cannot do even with help. That is, the ZPD refers to the space of tasks that the learner can do with guidance. ZPD is "the domain of transitions that are accessible to the [learner]" (p. 211). Vygotsky claimed that instruction need not wait for full readiness, but rather instruction can provide a motivation for extending a learner's intellectual reach:

> Instruction is not limited to trailing after development or moving stride for stride along with it. It can move ahead of development, pushing it further and eliciting new formations (p. 198).

Vygotsky sees classroom instruction in its ideal state as something that "forces [the learner] to rise above himself" (p. 213). By leveraging Vygotsky's idea, we can begin to think about how this relates to appropriateness of a mathematical task. ZPD concepts suggest that a task is maximally appropriate for a particular learner when it is located in that learner's ZPD—that is, when it is a task that the student can do but only with guidance. Tasks that a student can do without guidance may be less appropriate, because there is no purpose for the task other than to practice or build confidence. Similarly, a task that the student cannot do with any amount of guidance has no purpose. Note that this is a local criterion in the sense that it applies to one student at a time. When selecting a task for an entire class, the instructor must maintain a sense of each learner's ZPD at the time, and judge whether a task is likely to hit the "center of mass" of all those ZPDs.

The concept of the ZPD informs the aspects of appropriateness listed above that involve mathematical expertise and student readiness, among others.

Cognitive load theory

Sweller's (1998) **cognitive load theory** (CLT) proposes that cognitive tasks carry with them three different kinds of load on the learner's working memory:

- **Intrinsic load**, which refers to the irreducible difficulty that the task itself carries.
- **Extrinsic load**, which refers to extra difficulty that is placed on a task due to the way it is posed or delivered.
- **Germane load**, which refers to difficulty that helps learning by leading to the production of **schemas**, or organized patterns of knowledge.

According to CLT, learning is most effective when the task is best aligned with "human cognitive architecture" as described by Sweller (1998). Thus CLT gives a framework for judging on at least a partial level the appropriateness of a mathematical task. Namely, a mathematical task that minimizes extrinsic load while including germane load would be more appropriate for learning than would the same task with heightened extrinsic load.

Self-determination theory

Ryan (2000) describes **self-determination theory** (SDT) as pertaining to concepts of human motivation. While motivation to perform a mathematical task can be affected by one's expertise and readiness, motivation in return influences one's focus and level of effort expended on those tasks. SDT proposes that all learners possess three basic cognitive needs:

- The need for **competence**, which refers to expertise in performing tasks in a given context.
- The need for **autonomy**, which refers to the ability to locate the source of one's competence within oneself.
- The need for **relatedness**, which refers to a sense of belonging or association with a social group in a given context.

SDT also makes a distinction between **intrinsic motivation** and **extrinsic motivation**. Intrinsic motivation for a task refers to an internal, natural, inherent interest in the task, whereas extrinsic motivation refers to interest that is driven by forces outside the learners. For our purposes, the relevance of SDT to the selection or construction of appropriate mathematical tasks are:

- A task that increases intrinsic motivation is more appropriate for students' long-term intellectual development and short-term learning goals in a course than one that relies on extrinsic motivation.
- Intrinsic motivation is heightened when a task provides a feeling of competence, but only when accompanied by a sense of autonomy.
- At the same time, intrinsic motivation is more likely to flourish in social contexts that enhance a feeling of security and relatedness (Niemiec and Ryan, 2009).

Hence, for the purposes of selecting a mathematical task, an instructor should attend to whether a task promotes both competence and autonomy, as well as to whether the social environment of the class promotes relatedness.

CP.2.4. How to select an appropriate mathematical task

There is no formula for selecting an appropriate mathematical task, but the above considerations can guide instructors' choices. For example, an instructor could ask the following questions:

- Do I have clearly-stated and concrete learning objectives defined for the lesson in which the task is going to appear, and do students have access to those objectives?
- Does the task align with my learning objectives?
- Do I have actionable information, based on formative assessment or surveys, about my students' motivations, attitudes, and mathematical readiness for the task?
- Based on that information, does the task meet students at their level of expertise (not too easy, not too hard) and at their level of readiness (they are prepared to do the task apart from having the right level of expertise) and motivation (students have a reason to perform the task apart from extrinsic rewards and punishments)?
- Is the task well-constructed in terms of building students' intellectual development, competence, and autonomy? Does it leverage the social context of the class to promote relatedness?
- Is the task suitable for the physical environment of the class meeting?
- Is the task suitable for the mode of instruction (face-to-face vs. online)?

It seems unlikely that the answer to all of these questions will be "yes" for any single task, but at least these guidelines can help facilitate good design choices. The following section provides insights into choosing meaningful group-worthy tasks.

CP.2.5. Choosing meaningful group-worthy tasks

Stein, Grover, and Henningsen (1996) define a **mathematical task** as a set of problems or a single complex problem that focuses student attention on a particular mathematical idea. In this section we elaborate on **group-worthy tasks**, which provide opportunities for students to develop deeper mathematical meaning for ideas, model and apply their knowledge to new situations, make connections across representations and ideas, and engage in higher-level reasoning where students discuss assumptions, general reasoning strategies, and conclusions. In her chapter on "Crafting Group-worthy Learning Tasks", Lotan (2014, pp. 85–97) outlines a number of characteristics for academically challenging, intellectual, and rigorous tasks:

- They are open-ended, productively uncertain, and require complex problem solving.
- They provide opportunities for students to use multiple intellectual abilities to access the task and to determine intellectual competence.
- They address discipline-based, intellectually important content.
- They require positive interdependence and individual accountability.

Another way to think about characteristics of group-worthy tasks is in terms of the cognitive demand required. Stein et al. (1996) distinguish between low-level cognitive demand and high-level cognitive demand as shown in Table 1. Group-worthy tasks are characterized by high-level cognitive demand. One thing to notice in Table 1 is that high-level cognitive demand tasks can encompass both deep understanding about procedures and what may be thought of as relational or conceptual understanding. We find this useful because it moves beyond superficial arguments that often pit procedural knowledge against conceptual knowledge. The important thing is, as Lotan (2014, pp. 85–97) points out, that tasks require complex problem solving skills, use varied intellectual abilities, address significant mathematics, and require peer-to-peer interaction.

Before providing examples of group-worthy tasks, we quote Hsu, Kysh, and Resek's (2007, p. 7–8) depiction of classroom settings where such tasks are used.

- Students are placed into groups that are purposefully not homogenous. We commonly group students using some public random process, such as counting off, so it is clear there is no pre-grouping by perceived strength. For instructors who know the strengths and weaknesses of their students, we encourage purposeful grouping where students work collaboratively with students who have similar mathematical abilities.
- All students are given the same initial problem to work on in their non-homogenous groups. Groups who think they have finished early are asked to consider alternative methods of solution, further generalization, or other extension questions.
- Groups are expected to be responsible for the respectful learning of all members.
- Students need to communicate their reasoning so others can understand and build on it. Usually students are expected to share their work with the class, either in informal whole class discussion or as formal presentations, which sometimes include prepared overheads or posters.
- Students are expected to struggle with the problem and to negotiate and argue their different ideas.
- Good mathematical argument and explanation are emphasized as goals and are necessary to the reporting process.
- Creative approaches are encouraged and analyzed, even if they don't lead directly to a solution.

We next analyze a group-worthy, cognitively demanding task and discuss how the task can provoke mathematical curiosity, struggle, discussion about mathematics, and insight into important ideas. This task is part of a research-based inquiry-oriented curriculum developed by Rasmussen and colleagues.

<table>
<tr><th>Low-level cognitive demands</th><th>High-level cognitive demands</th></tr>
<tr><td>

Memorization tasks

- Involve either producing previously learned facts, rules, formulae, or definition, *or* committing facts, rules, formulae, or definitions to memory.
- Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure.
- Are not ambiguous—such tasks involve exact reproduction of previously seen content and what is to be reproduced is clearly and directly stated.
- Have no connection to the concepts or meaning that underlay the facts, rules, formulae, or definitions being learned or reproduced.

Procedures without connections tasks

- Are algorithmic. Use of the procedure is either specifically called for or its use is evident based on prior instruction, experience, or placement of the task.
- Require limited cognitive demand for successful completion. There is little ambiguity about what needs to be done and how to do it,
- Have no connection to the concepts or meaning that underlie the procedure being used.
- Are focused on producing correct answers rather than developing mathematical understanding.
- Require no explanations or require only explanations that focus solely on describing the procedure that was used.

</td><td>

Procedures with connections tasks

- Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas.
- Suggest pathways to follow (explicitly or implicitly) that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts.
- Usually are represented in multiple ways (e.g., visual diagrams, manipulatives, symbols, problem situations). Making connections among multiple representations helps to develop meaning.
- Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with the conceptual ideas that underlie the procedures in order to successfully complete the task and develop understanding.

Doing mathematics tasks

- Require complex and non-algorithmic thinking (i.e., there is not a predictable, well-researched approach or pathway explicitly suggested by the task instructions, or a worked-out example).
- Require students to explore and to understand the nature of mathematical concepts, processes, or relationships.
- Demand self-monitoring or self-regulation of one's own cognitive processes.
- Require students to access relevant knowledge in working through the task.
- Require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions.
- Require considerable cognitive effort and may involve some level of anxiety for the student due to the unpredictable nature of the solution process required.

</td></tr>
</table>

Table 1. Low- and high-level cognitive demands (Stein et al., 1996).

Classroom vignette: The fish.net task

This task can be used as a capstone for students' study of first order autonomous differential equations (DEs). The task presents an opportunity for students to critique given mathematial models, use and connect multiple representations, and communicate their results and reasoning to others. It also provides an opportunity to reinvent a bifurcation diagram, although doing so is not a necessary component of a complete solution.

This task is intended for classrooms that allow students to work together in small groups and present their solutions to others in the class. The task, referred to as the "Fish.net task", has three main parts. The first two parts are intended to be completed in class. The third part is intended to be started in class, but then significant out-of-class work is expected. Prior to implementing this task, students should have studied various graphical, analytical, and numerical techniques for first order DEs in general and for autonomous DEs, in particular. There is no expectation that students are familiar with bifurcation diagrams. In fact, it is best if they have never seen such diagrams.

The task: A mathematician at a fish hatchery has been using the differential equation $\frac{dP}{dt}=2P\left(1-\frac{P}{25}\right)$ as a model for predicting the number of fish that a hatchery can expect to find in its pond. Use a graph of $\frac{dP}{dt}$ vs. P, a phase line, and a slope field to analyze what this differential equation predicts for future fish populations for a range of initial conditions. Present all three of these representations and describe in a few sentences how to interpret them.

Recently, the hatchery was bought out by fish.net and the new owners are planning to allow the public to catch fish at the hatchery (for a fee of course). This means that the previous differential equation used to predict future fish populations needs to be modified to reflect this new plan. For the sake of simplicity, assume that this new plan can be taken into consideration by including a constant annual harvesting rate in the previous differential equation. Which of the three modified differential equations makes the most sense to you and why?

(a) $\frac{dP}{dt}=2P\left(1-\frac{P}{25}\right)-kp$ (b) $\frac{dP}{dt}=2P\left(1-\frac{P-k}{25}\right)$ (c) $\frac{dP}{dt}=2P\left(1-\frac{P}{25}\right)-k$

Using the modified differential equation agreed upon from the previous problem, prepare a one page report for the new owners that illustrates the implications that various choices of will have on future fish populations. Your report may include one or more graphical representations but must synthesize your analysis of the effect of different values in a concise way.

The first part of the fish.net task provides an opportunity for students to use and connect multiple representations and to interpret how these various representations tell the story of how the differential equation predicts future population values for different initial conditions. Making connections is a characteristic of cognitively demanding, group-worthy tasks. Together the three parts of the task have a "low floor" and a "high ceiling", meaning that all students should be able to make significant and meaningful progress on the first task, while the third task provides an opportunity for students to excel and essentially reinvent a bifurcation diagram. The reinvention of new mathematics is another characteristic of a cognitively demanding, group-worthy task.

In the second part of the task, students productively struggle to determine which modification to the DE makes the most sense and present their reasoning to the class. Typically, the class agrees that option c) makes the most sense, but getting to that point requires struggle, negotiation, and sharing their reasoning. Having settled on option c) for the modified differential equation, the students move on to the third part of the task, where they analyze how a change in a parameter affects the space of solutions that predict how the fish population will unfold over time for different initial conditions. Students work in small groups during class and work with their group outside of class to finish the analysis and prepare a one-page summary report. This combination of in and out of class group work promotes positive interdependence and individual accountability, which is another important characteristic of group-worthy tasks.

The rationale for constraining the final report to one page is to encourage students to be creative in ways

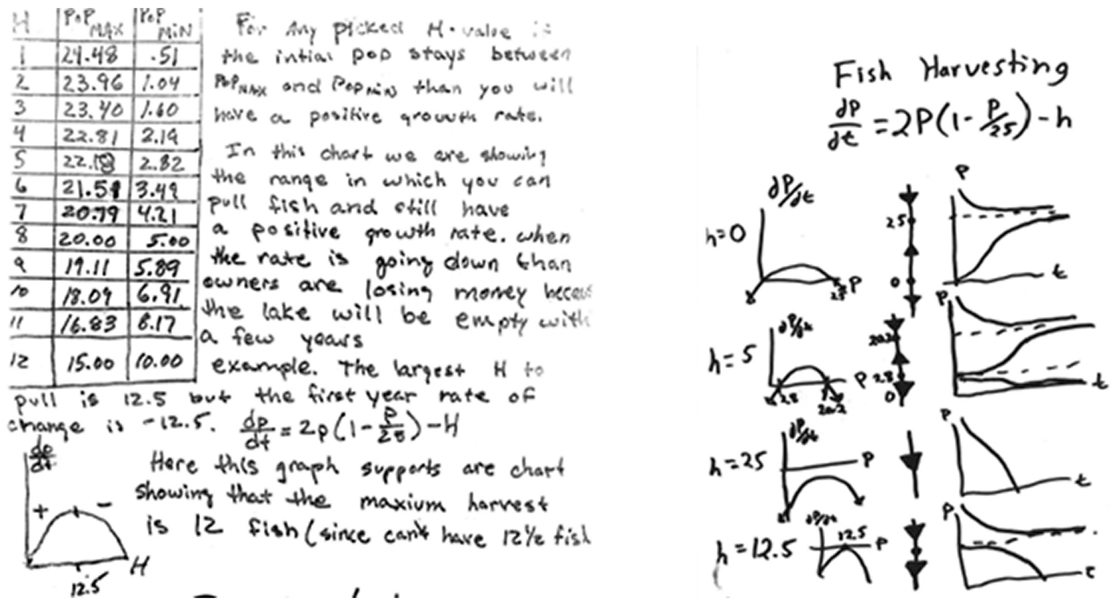


Figure 6. Group 1 and Group 2 fish.net summary reports.

to represent their analysis of multiple k values in a concise manner. Figure 6 shows two examples of summary reports from different groups of students.

In the first report students summarize their analysis using a table and paragraph detailing how to interpret the table. The second group's report made use of multiple, linked graphical representations to illustrate the effect of changing the parameter on solutions to the differential equations. Having different types of reports provides students an opportunity to make connections across reports. For example, students might discuss how the columns in the table presented by the first group relate to the graphs presented by the second group.

The next example illustrates another creative way in which students synthesized their analysis and resulted in a bifurcation diagram. In this presentation the students leveraged an additional representation in their analysis by actually finding an equation for the equilibrium solutions as a function of the parameter k. Determining the usefulness of the equation, carrying out the algebraic steps, and graphing the result can be non-trivial for students and represents important mathematical connections and skills. Such mathematical work reflects aspects of cognitively demanding tasks that highlight "procedures with connections" (Stein et al., 1996).

The reader might be surprised to learn that the first three group reports all came from the same class. In the midst of the third group presenting their analysis, students figured out that they could "drop a phase line" on the graph of P vs k, thereby making further connections across group reports. A screenshot of a student having done this is shown in Figure 7(b). This third report provides additional opportunities for students to make connections across the different small group reports. Moreover, the student-generated graph of P vs k provides an opportunity for the instructor to label student work with the terms bifurcation and bifurcation diagram, not only formalizing the mathematical work of students but also providing students with a sense of mathematical ownership and significant accomplishment.

Finally, in the fourth group report (from a different class), students adopted creative use of *Excel*, where the rows are P values, columns are k values, and the cells contain dP/dt values. Here again we see students rising to the challenge of producing creative and original analyses that address significant and deep mathematical ideas and with the aid of technology.

Reflecting on the nature of the fish.net task, we revisit Lotan's (2014, pp. 85–97) characteristics of group-worthy tasks and find that this task satisfies all four characteristics.

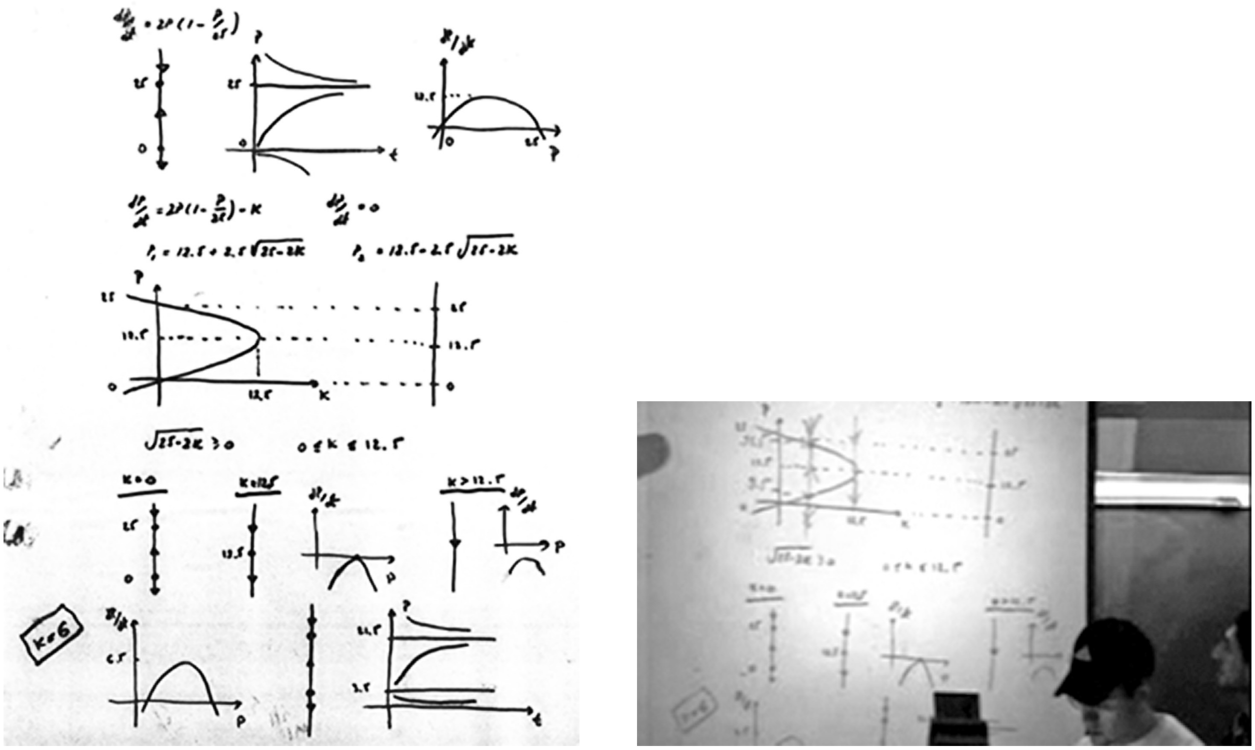


Figure 7. (a) Fish.net report from group 3 (b) Result of student "dropping" a phase line onto the *P* vs *k* graph.

2P(1-P/25)

P	dP/dt	1	2	3	4	5	6	7	8	9	10	11	12	12.5
1	1.92	0.92	-0.08	-1.08	-2.08	-3.08	-4.08	-5.08	-6.08	-7.08	-8.08	-9.08	-10.08	-10.58
2	3.68	2.68	1.68	0.68	-0.32	-1.32	-2.32	-3.32	-4.32	-5.32	-6.32	-7.32	-8.32	-8.82
3	5.28	4.28	3.28	2.28	1.28	0.28	-0.72	-1.72	-2.72	-3.72	-4.72	-5.72	-6.72	-7.22
4	6.72	5.72	4.72	3.72	2.72	1.72	0.72	-0.28	-1.28	-2.28	-3.28	-4.28	-5.28	-5.78
5	8	7	6	5	4	3	2	1	0	-1	-2	-3	-4	-4.5
6	9.12	8.12	7.12	6.12	5.12	4.12	3.12	2.12	1.12	0.12	-0.88	-1.88	-2.88	-3.38
7	10.08	9.08	8.08	7.08	6.08	5.08	4.08	3.08	2.08	1.08	0.08	-0.92	-1.92	-2.42
8	10.88	9.88	8.88	7.88	6.88	5.88	4.88	3.88	2.88	1.88	0.88	-0.12	-1.12	-1.62
9	11.52	10.52	9.52	8.52	7.52	6.52	5.52	4.52	3.52	2.52	1.52	0.52	-0.48	-0.98
10	12	11	10	9	8	7	6	5	4	3	2	1	0	-0.5
11	12.32	11.32	10.32	9.32	8.32	7.32	6.32	5.32	4.32	3.32	2.32	1.32	0.32	-0.18
12	12.48	11.48	10.48	9.48	8.48	7.48	6.48	5.48	4.48	3.48	2.48	1.48	0.48	-0.02
13	12.48	11.48	10.48	9.48	8.48	7.48	6.48	5.48	4.48	3.48	2.48	1.48	0.48	-0.02
14	12.32	11.32	10.32	9.32	8.32	7.32	6.32	5.32	4.32	3.32	2.32	1.32	0.32	-0.18
15	12	11	10	9	8	7	6	5	4	3	2	1	0	-0.5
16	11.52	10.52	9.52	8.52	7.52	6.52	5.52	4.52	3.52	2.52	1.52	0.52	-0.48	-0.98
17	10.88	9.88	8.88	7.88	6.88	5.88	4.88	3.88	2.88	1.88	0.88	-0.12	-1.12	-1.62
18	10.08	9.08	8.08	7.08	6.08	5.08	4.08	3.08	2.08	1.08	0.08	-0.92	-1.92	-2.42
19	9.12	8.12	7.12	6.12	5.12	4.12	3.12	2.12	1.12	0.12	-0.88	-1.88	-2.88	-3.38
20	8	7	6	5	4	3	2	1	0	-1	-2	-3	-4	-4.5
21	6.72	5.72	4.72	3.72	2.72	1.72	0.72	-0.28	-1.28	-2.28	-3.28	-4.28	-5.28	-5.78
22	5.28	4.28	3.28	2.28	1.28	0.28	-0.72	-1.72	-2.72	-3.72	-4.72	-5.72	-6.72	-7.22
23	3.68	2.68	1.68	0.68	-0.32	-1.32	-2.32	-3.32	-4.32	-5.32	-6.32	-7.32	-8.32	-8.82
24	1.92	0.92	-0.08	-1.08	-2.08	-3.08	-4.08	-5.08	-6.08	-7.08	-8.08	-9.08	-10.08	-10.58
25	0	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10	-11	-12	-12.5
26	-2.08	-3.08	-4.08	-5.08	-6.08	-7.08	-8.08	-9.08	-10.08	-11.08	-12.08	-13.08	-14.08	-14.58
27	-4.32	-5.32	-6.32	-7.32	-8.32	-9.32	-10.32	-11.32	-12.32	-13.32	-14.32	-15.32	-16.32	-16.82
28	-6.72	-7.72	-8.72	-9.72	-10.72	-11.72	-12.72	-13.72	-14.72	-15.72	-16.72	-17.72	-18.72	-19.22
29	-9.28	-10.28	-11.28	-12.28	-13.28	-14.28	-15.28	-16.28	-17.28	-18.28	-19.28	-20.28	-21.28	-21.78
30	-12	-13	-14	-15	-16	-17	-18	-19	-20	-21	-22	-23	-24	-24.5

Figure 8. Use of *Excel* to represent analysis of fish.net task.

- Part 3 is open-ended and requires complex problem solving.
- All three parts provide opportunities for students to use multiple intellectual abilities to access the task.
- The task as whole addresses discipline-based, intellectually important content such as bifurcation theory and understanding model parameters.
- The group reports require positive interdependence and individual accountability.

To conclude, we note that once the ideas and terminology of bifurcation and bifurcation diagram have been appropriately connected to student work and defined, students are now in a position to follow up on the fish.net task with problems that have no application context and that serve to illustrate different types of bifurcations. Below are some suitable follow-up tasks. The possibility of such follow-up tasks is another indicator that the fish.net task contains important mathematics and has a high ceiling.

1. For each of the following, develop a report that illustrates (with a suitable graph or graphs) and describes (in words) the way in which the solutions change as the value of r changes. Identify the precise value(s) of r for which there is either a change in the number of equilibrium solution(s) or a change in the type of equilibrium solution(s).

   a) $\frac{dy}{dt} = (y-3)^2 + r$ b) $\frac{dy}{dt} = y^2 - ry + 1$ c) $\frac{dy}{dt} = ry + y^3$ d) $\frac{dy}{dt} = y^6 - 2y^4 + r$

2. For part a) in problem 1, sketch a graph of the equilibrium solutions as r varies. Such a graph is referred to as "bifurcation diagram."
3. For part b) in problem 1, sketch a graph of the equilibrium solutions as r varies. Such a graph is referred to as "bifurcation diagram."

CP.2.6. Communication: Reading, writing, presenting, visualizing

In 2009, state leaders from around the United States began development of what we now know as the *Common Core State Standards* (CCSS; NGA and CCSSO, 2010), which include both mathematics standards and English Language Arts (ELA) standards. These state leaders had a common interest to create, as stated in the current CCSS documents, a rigorous set of standards that prepare all students for college and career. "State school chiefs and governors recognized the value of consistent, real-world learning goals and launched this effort to ensure all students, regardless of where they live, graduating high school prepared for college, career, and life." (www.corestandards.org/about-the-standards/development-process/)

In their current form, the CCSS (NGA and CCSSO, 2010) for mathematics have two major parts: *Content Standards* and *Standards for Mathematical Practices*. While the content standards might look more familiar to most people, they were designed with learning progressions in mind, with a focus on student understanding of mathematical ideas, and with attention to applications of mathematical ideas in real-world contexts. The Standards for Mathematical Practices, however, "describe varieties of expertise that mathematics educators at all levels should seek to develop in their students." The goal is to hold students accountable for

1. Making sense of problems and perseverance in solving them.
2. Reasoning abstractly and quantitatively.
3. Constructing viable arguments and critiquing others' reasoning.
4. Modeling with mathematics.
5. Using appropriate tools strategically.
6. Attending to precision.
7. Looking for and making use of structure.

8. Looking for and expressing regularity in repeated reasoning.

Math Practice #3 provides an excellent vision of what should be happening in college mathematics classrooms where profound student learning is the goal.

Construct viable arguments and critique the reasoning of others

There has long been a focus on asking students to follow procedures and to master algorithms often without conceptual understanding. Research suggests that students learn mathematics by doing mathematics, which includes a balance between conceptual understanding, procedural fluency, and applications (Shellard and Moyer, 2002). Additionally, when procedural fluency is necessary, it is developed through and emerges from conceptual understanding and, perhaps, applications. In other words, the teaching of mathematical concepts and skills should be centered around problems to be solved (Checkly, 1997; Wood and Sellars, 1996; Wood and Sellars, 1997). It is important for students to develop mathematical meaning to address the challenges we currently observe in mathematics education in the United States. If we do not focus on mathematical meaning, "the result is teachers' inability to teach for understanding and students' inability to develop personal mathematical meanings that support interest, curiosity, and future learning" (Thompson, 2013, p. 57). A recent *Programme for International Student Assessment* report (Piacentini and Monticone, 2016) from the Organisation for Economic Cooperation and Development (OECD) confirms the importance of students developing foundational understanding of mathematical ideas. Andreas Schleicher, OECD Director of Education and Skills summarized the recent results:

> Our analysis is [that] when students have really understood the foundations, they can extrapolate. They can apply that knowledge in another context. However, if they only teach students tips and tricks, how to solve small everyday problems, they know how to solve those problems, but they're not good at transferring that knowledge to another context (Barshay, 2016).

As in previous discussions, constructing viable arguments entails students engaged in problem solving tasks (a mathematical problem or a real-world context) and asked to articulate their reasoning as they demonstrate their solution to the problem. Students should come to interpret the word "viable" as "possible" so that during presentations students recognize that they are considering a possible solution that requires analysis in order to determine its mathematical worthiness. Students may work independently or collaboratively to create their viable argument and the argument can be delivered in written or spoken form. To demonstrate the idea of making a viable argument, consider the following problem.

The Playground Problem: Students at a school are told that the principal plans to double the length of each side of a square playground. The students are pleasantly surprised to discover the impact this has on the area of the playground.

Solution #1—Algebraic response

In the solution shown in Figure 9, the student's focus was on making algebraic representations of the necessary quantities. Through this symbolic representation of the diagram of the playground, a student can make a viable argument about the size of the playground after doubling the length of the sides. The class can then critique the reasoning of this argument by pointing out aspects of the solution that make sense, aspects that one might question, or aspects that one might improve or edit with appropriate discussion.

Solution #2—Visual response

In the solution shown in Figure 10, we see evidence of student reasoning about the area of the playground

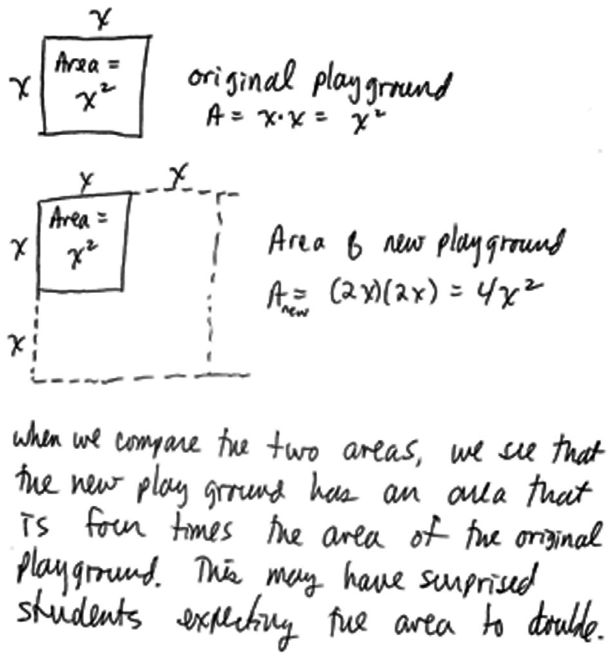


Figure 9. Algebraic response to Playground problem.

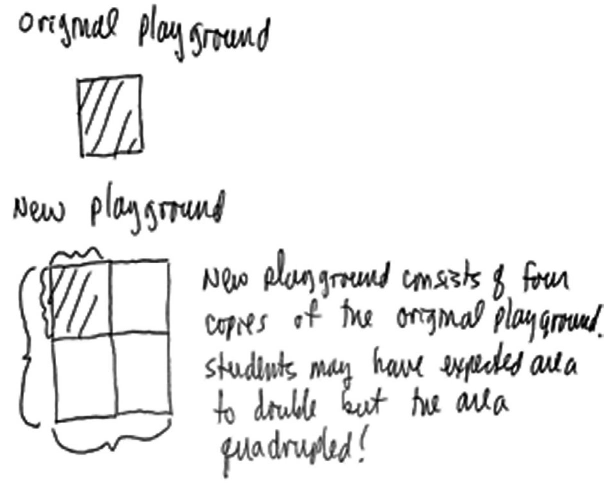


Figure 10. Visual response to Playground problem.

after doubling the length of the sides. Again, imagine a student presenting their reasoning with this solution on display. Certainly, there are ways to improve the argument (more labels) but there are also very powerful images that capture the essence of the solution, as we can see the four original squares in the new playground.

The second component of the third mathematical practice is critiquing other students' work, which entails listening, evaluating, and providing feedback on the reasoning of others. This can be observed in many ways, some of which will be described.

Sometimes, students work in groups of 2–4 students to solve a problem, develop a mathematical idea, or create and defend a mathematical conjecture. In this small group context students often are afforded the opportunity to listen to one another and critique the reasoning of others. This happens naturally as students share ideas, articulate solution paths, or back up any claims made. Critiquing the reasoning of others can also be observed in a whole group setting. A skilled instructor may ask any number of students to present their work and reasoning to the entire class. Students can be selected to share based on the fact that their work

- Offers a unique approach.
- Provides context to discuss an important idea.
- Showcases outstanding work.
- Demonstrates solutions focusing on different mathematical representations.

Often, people view the idea of a critique as something meant to expose an error, but in the mathematics classroom it

- **Confirms the work of another student:** Rather than simply saying, "That makes sense", students should articulate clearly what specifically makes sense to them.
- **Questions the work of another student:** It can be the case that a member of the audience is unclear about some portion of the presentation and should inquire for the purpose of gaining clarity.

- **Disagrees with the work of another student:** If a student thinks that the presentation is in error, they should respectfully discuss this with the presenting student for the purpose of unpacking the truth.

Stein and Smith (2011) provide a framework for instructors to improve classroom discourse by focusing on student development of understanding mathematical ideas. Prior to implementing a well-designed classroom task, instructors spend time **anticipating** student responses by working through the task independently and thinking about multiple solution strategies. For each strategy instructors should consider how they will respond to what students produce during the lesson. During the lesson instructors **monitor** students' work on, and engagement with, the task. Instructors tour the classroom, listen to student thinking, ask probing questions, and encourage student thinking and sharing. As instructors monitor they watch for opportunities to **select** student work (individual or group) to be presented to the entire class. Student work is selected based on the instructional and mathematical goals of the lesson. Interesting work is selected to show a variety of solution strategies, common mistakes, or unique ways of thinking. Selected students communicate their thinking (viable argument) and allow others to critique their reasoning. Prior to presenting, the instructor must **sequence** students' responses. The sequence is well thought out and intentional to tell a story or maybe to show less sophisticated to more sophisticated reasoning. Sometimes a sequence is designed to show common mistakes first and mathematically sound solutions later (or vice versa) depending on the learning goals at the moment. Finally, instructors work to help students make **connections** between the different student responses as appropriate. Sometimes, solution strategies are mathematically equivalent and students should be able to recognize and articulate these equivalencies. Sometimes, solution strategies demonstrate a variety of mathematical representations and students should make the connections between these representations.

Additionally, instructors must think about the nature of the selected tasks that will promote positive classroom discourse. If the mathematical focus of the task is following a prescribed procedure for the purpose of producing right answers, the classroom discourse could be limited and shallow. On the other hand, if the mathematical focus of the task is appropriately challenging and interesting to students, there can be great potential for rich discourse and powerful learning.

CP.2.7. Error analysis of student work

Engaging in mathematical tasks often results in errors by students. Recent research has shown that our brains grow when we make mistakes (Moser et al., 2011). That is, making mistakes is one of the most important things to do for mathematical growth. Anecdotally, we know that the more we attempt to reach new goals, the more likely we are to fail in the short term but the more likely we are to succeed in the longer term. In mathematics, this means we need to find ways to embrace mistakes and retask them as learning opportunities, both for us as instructors and for our students.

Error analysis in mathematics, that is, the close examination of incorrect mathematical student work has been a focus of instructors and researchers for years, across mathematical contexts and in a variety of countries (e.g., Ganesan and Dindyal, 2014, pp. 231–238; Kingsdorf and Krawec, 2014; Luneta and Makonye, 2010). Closely examining students' incorrect work can help us as instructors by providing a real context in which we can explore students' thought processes and use that work to diagnose misconceptions, usually with a goal of remediation. By examining actual student work we can differentiate between what Olivier (1989) refers to as **slips**, which are results of carelessness, **errors**, which are systematic reflections of student misconceptions, and **misconceptions**, the underlying incorrect beliefs students hold with respect to the mathematical processes.

In addition to instructor examination of errors, a more recent trend involves having students examine

Incorrect definition

Math content:

What properties can be derived from this definition?
Which ones fit our image of the concept? Which ones don't?
What other mathematical objects could be described by this definition?
What instances of the concept are not described by this definition?
Are all the properties stated essential? Could any be eliminated?
Could we modify the definition and turn it into a correct one?
What if this were the correct definition for the concept?
What would the concept itself be?
How would it compare with the standard one?
What would be the consequences of accepting this definition in mathematics?
How could this definition be further modified?
What other alternative notions could be created?

Nature of mathematics:

What characteristics do we want a mathematical definition not to have? What properties should a mathematical definition have?
How can we evaluate and choose among alternative definitions for a given concept?
What should a definition accomplish? What do we use definitions for?
How do mathematical definitions differ from those in other fields?

Approximate results

Math content:

Can you evaluate how "big" an error you are making by using the approximate result instead of the exact one?
What would be the consequences of your "error", once you use the approximate result in other applications?
Are other approximate results available?
How do they compare with yours?
Could you further improve your result. and obtain a "closer" approximate result?
Would such an activity be worth it?

Nature of mathematics:

Can we always get exact results in mathematical problems?
If not why?
What could the role and value of approximate results be when exact results are available?
What if the exact results are not available?
How can alternative approximate results be evaluated?

Wrong results

Math content:

In what sense is the result wrong?
Where did the procedure fail?
Could it be fixed up and thus lead to different results?
What were our assumptions and are they justified?
In what cases?
What are the consequences of accepting this alternative result?
In what circumstances could such a result be considered right?

Nature of mathematics:

How can we test whether we used a mathematical procedure correctly?
How can we decide whether it is appropriate to apply a certain procedure in a given situation?
How can we determine the domain of application of a given procedure?

Right results reached by an unsatisfactory procedure

Math content:

Why do we get right results in this case?
Could the procedure be slightly modified and be made more rigorous?
Does the procedure work in this specific case because of specific properties pertaining to it?
In such case what are these properties?
In what cases would it work?
In what cases would it fail?
What assumptions are necessary to be sure it will work?

Nature of mathematics:

Is the difference between being rigorous or not rigorous a difference in degree?
Who decides whether a procedure is sufficiently rigorous?
On what basis?
Were the criteria used the same throughout the history of mathematics?

Unsatisfactory models

Math content:

In what sense does the model work and in what sense does it not?
How does the model compare with another "good" model of the same· concept if there are any such models available?
Why does the model fail to represent some aspects of the concept?
How could we try to modify the model so that it "fits" the concept better?
Is the real problem a limitation in the specific model or in the concept itself?

Nature of mathematics:

How can we determine the aspects for which a model "fits'" the original object and the aspects for which it does not fit?
How "different" from the actual object could an acceptable model be?
What is the value of alternative yet impartial models?
How could we evaluate which one is better?

Table 2. Questions generated by some common types of errors (Borasi, 1987, p. 6).

incorrect student work (real or instructor-created) to explore mathematics more deeply and refine their own thinking. Borasi (1987, 1994) calls errors "springboards for inquiry" and puts forward a strategy for using them to stimulate worthwhile mathematical inquiries. Errors can be used to investigate the nature of fundamental mathematical notions such as "proof", "algorithm" and "definition". For example, instructors might provide students with slightly incomplete or incorrect proofs (or algorithms or definitions) and have them find the point of error. From these processes, students can begin to formalize what constitutes a "good" proof. Table 2 illustrates potential errors that occur during inquiry and provides questions that instructors can ask to help the student decipher the error or to facilitate a classroom discussion. The nature of mathematics questions can be helpful in redesigning lessons or activities that promote classroom discourse.

However, deciding to use student errors in our teaching requires some thought. The choice of tasks through which our students do and learn mathematics is critical in ensuring we provide a growth opportunity through some sort of error analysis. How do we choose and structure mathematical tasks that elicit the student thinking we want to see? Boaler (2015) gives six suggestions for developing or adapting mathematical tasks to increase their potential for providing open learning spaces. They include

- Open the task up to multiple methods, pathways and representations.
- Include inquiry opportunities.
- Ask the problem before teaching the method.
- Add a visual component and ask students how they see the mathematics.
- Extend the task to make it lower floor and higher ceiling.
- Ask students to convince and reason—be skeptical.

When we choose or create tasks we must first decide who will use them. Are they for individual instructors, a group of instructors, individual students, students working in groups, etc.? Below are some examples of ways to use student errors to improve our assessment of students' reasoning as well as to improve their understanding of mathematical constructs.

Carlson, Oehrtman, and Engelke (2010) designed the assessment shown in Figure 11 with purposeful distractors that were created based on student misconceptions as documented in research. It might surprise readers to learn that only 25% of 672 precalculus students chose the correct answer.

Which of the following formulas defines the area, *A*, of a square in terms of its perimeter, *p*?

a) $A = \frac{p^2}{16}$

b) $A = s^2$

c) $A = \frac{p^2}{4}$

d) $A = 16x^2$

e) $p = 4\sqrt{A}$

Figure 11. Precalculus item from Carlson, Oehrtman, and Engelke, 2010.

Exploring the sources of incorrect answers can inform instructors where we might need to intervene, clarify, and remediate. This exploration can be accomplished with open-ended questions, group-discussion tasks, or reflective writing assignments centered on a particular, common incorrect answer.

CP.2.8. Flipped classrooms

So far our discussion of the selection of appropriate mathematical tasks has made tacit assumptions about the structure of the course for which those tasks are selected, namely:

- Students are working in one of two contexts: individually or collaboratively. We refer to the contexts in which students work individually as their **individual space** and the context in which students work in structured, managed groups as the **group space** for the course. For a typical face-to-face course, the group space is the class meeting and the individual space is everything in between.
- Students meet in the group space for the purpose of gaining first contact with new ideas and engaging in some preliminary explorations of the analysis and applications of those new ideas.
- Students work in their individual spaces to explore higher-order cognitive tasks such as advanced applications, synthesis of new ideas with old ones, and creative tasks such as the writing of proofs or construction of models.

We refer to a course design that uses individual and group space in these ways as a **traditionally designed** course. The appropriateness of a mathematical task depends on the choice of design and the context in which it is intended to be used. For example, a task that asks students to write a proof of a conjecture might not be appropriate for the individual space if the students have only just encountered the concept for the first time. Instead, an activity having students explore the concept that leads to a formulation of the conjecture seems more appropriate, followed by assigning the proof for the individual space.

However, an increasing number of instructors are choosing to employ a **flipped learning** design in their courses. Flipped learning is a pedagogical model in which first contact with new ideas takes place in the individual space rather than in the group space, and the group space is repurposed to focus on active learning and creative applications of those ideas. The above task in which students explore an idea and make a conjecture might be more appropriate for the individual space in a flipped learning environment, and then students come to the group space with their conjectures and work together to construct and critique proofs of that conjecture. The appropriateness of a given mathematical task may change in a flipped course model.

What kinds of tasks are appropriate for flipped learning environments? To address this question, consider that the flipped learning environment proceeds as a cycle through several phases:

1. The individual space prior to the group meeting.
2. The first few minutes of the group meeting.
3. The main portion of the group meeting.
4. The last few minutes of the group meeting.
5. The ongoing individual space following the group meeting.

Each phase has certain tasks that are highly appropriate for that particular context.

Prior to group meetings, student work in a flipped learning environment focuses on gaining first contact with new concepts through the use of structured activities. Ideally, students should gain basic fluency with new ideas so that in the group space they can work together on advanced, creative work with those ideas without extensive review.

Therefore the most appropriate tasks to select for pre-group meeting work should focus on the **lowest levels of Bloom's taxonomy** (Anderson, et al., 2001) the storage and retrieval of basic facts and concepts ("Remembering"), the explanation of those ideas ("Understanding"), and using those ideas in simple new situations ("Applying" on a basic level). Tasks at this level are appropriate because this phase of the learning process is the student's first contact with new material. To keep student motivation high, anything beyond these levels should be saved for later. Moreover, the ways in which these tasks are presented must be carefully selected. Students are asked to pick up new concepts—some of which may be quite complex—through their own efforts. Flipped learning makes no apologies for this, but at the same time recognizes that scaffolding and guidance is needed in order to maximize the likelihood of success and to maintain reasonable levels of motivation.

CP.2.9. Procedural fluency emerges from conceptual understanding

Procedural fluency and how it is developed is of central importance to post-secondary mathematics and has been discussed for many years in the K–12 mathematics education community. Many teachers at all levels of K–12 education lament the lack of basic skills—especially multiplication facts—exhibited by many students. In 2014 the National Council of Teachers of Mathematics (NCTM) released a position paper on the issue and defined procedural fluency as follows (2014):

> Procedural fluency is a critical component of mathematical proficiency. Procedural fluency is the ability to apply procedures accurately, efficiently, and flexibly; to transfer procedures to different problems and contexts; to build or modify procedures from other procedures; and to recognize when one strategy or procedure is more appropriate to apply than another. To develop procedural fluency, students need experience in integrating concepts and procedures and building on familiar procedures as they create their own informal strategies and procedures. Students need opportunities to justify both informal strategies and commonly used procedures mathematically, to support and justify their choices of appropriate procedures, and to strengthen their understanding and skill through distributed practice.

What are the areas of procedural fluency necessary for collegiate mathematics? Does the availability and accessibility of technology influence this discussion? If procedural fluency is desired, how is it developed? The first question is beyond the scope of this document and is best left for a discussion at the local level where colleagues can work collaboratively to make decisions that are best for their students and community. While having these discussions, instructors can also consider the second question—the role of technology in developing procedural fluency. The role of technology in the collegiate mathematics classroom is discussed elsewhere in this document. However, once the specific areas of desired procedural fluency are determined (e.g., solving a variety of equation types, use of function notation, differentiation and integration skills), there are general strategies to help students to develop procedural fluency.

Build procedural fluency from conceptual understanding

Conceptual understanding and procedural fluency can work hand in hand to help students to make sense of important mathematical ideas and to develop effective and productive problem-solving skills. Expressed simplistically, **conceptual understanding** involves knowing what to do and why it works, while **procedural fluency** involves deciding and knowing how to do it. While conceptual understanding and procedural fluency can work together as students engage in mathematical activity or in solving problems, we will focus on how procedural fluency develops from conceptual understanding.

Brain research often reports the importance of building strong cognitive connections. When students learn procedures in such a way that they are connected to conceptual foundations, they will have more success in using these procedures, will recall them for a longer period of time, and will be able to use these procedures flexibly and effectively in a problem solving situation (NRC, 2005). Consider fraction division as a case in point. Often students are told to "keep the first fraction, change the division sign to multiplication, and flip the second fraction." That is, they "keep-change-flip," and multiply the fractions to get the desired result. This technique can, in the short term, help students get the right answer to a computational problem. However, since this technique is not connected to the meaning of division or to an understanding of what a fraction is, students often forget the procedure or apply an incorrect procedure, also void of conceptual understanding, when faced with a fraction division situation. Furthermore, this "keep-change-flip" teaching strategy does not prepare students to understand situations where the division of fractions is even necessary or required. We see the impact of this lack of conceptual understanding in the collegiate classroom when students employ a remembered procedure at the wrong time. Without a foundation in conceptual understanding, students will grab at the procedures that they remember in hopes that it will produce the correct result (NRC, 2005).

Students can develop conceptual understanding, procedural fluency, and problem-solving strategies when they learn mathematics with a focus on the *Standards for Mathematical Practices* (CCSS) and when teachers effectively implement the *Mathematics Teaching Practices* (NCTM, 2014). Bullmaster-Day (n.d.) summarizes the benefits of learning mathematics when the *Mathematical Practices* and *Mathematics Teaching Practices* are a major focus. She claims

> A consistent instructional cycle that incorporates all of these elements enables students to organize, store, and retrieve new knowledge, while strengthening interconnections between the pieces of information in their mental "maps" so that the information will be available to them for recall, transfer, and future use. When students have opportunity to practice skills to the point of automaticity their working memory is freed for new tasks and they are able to see patterns, relationships, and discrepancies in problems that they would have missed without such practice (Anderson, Greeno, Reder, and Simon, 2000; Bransford, Brown, and Cocking, 2000; Collins, Brown, and Newman, 1989; Ellis and Worthington, 1994; Good and Brophy, 2003; Marzano, Gaddy, and Dean, 2000; Means and Knapp, 1991; Pressley, et al., 1995; Rosenshine, 2002; Rosenshine and Meister, 1995; Stevenson and Stigler, 1992; Wenglinsky, 2002, 2004).

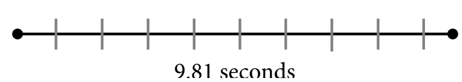
Classroom vignette: Average rate of change

The idea of average rate of change is one that is developed in collegiate algebra and precalculus courses and used later in courses such as calculus and differential equations. Ideally students become procedurally fluent in calculating the average rate of change in context, and conceptually make sense of what average rate of change means. In fact, knowing the meaning of the average rate of change can help students to compute it effectively and accurately.

A College Algebra class considers the situation from the 2016 Rio de Janeiro Olympics where Jamaican sprinter Usain Bolt won the gold medal in the 100-meter sprint with a time of 9.81 seconds. Students are asked to consider the speed at which Usain Bolt ran. It is correct to say that Usain Bolt ran the race at an average speed of 100 meters per 9.81 seconds, though no one would suggest this non-conventional way to express the idea of speed. When asked to describe the average speed, students might try to remember a formula. Even if they do so accurately, they cannot articulate mathematical reasons for why division is required.

Thinking Stage 1

Imagine that the segment represents 9.81 seconds and is cut up into a total of 9.81 one-second segments. Each segment represents $\frac{1}{9.81}$ of the total time.

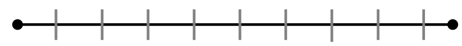


In one approach students are encouraged to think that the goal is to report Usain Bolt's speed as a constant speed per one second (a unit rate). There are 9.81 one-second intervals contained in the total time of 9.81 seconds. Each of these one-second intervals represents $\frac{1}{9.81}$ of the total time of 9.81 seconds.

In order to maintain the proportional relationship between distance and time, the 100-meter distance must also be cut up into 9.81 segments so that we can claim that each of these distance segments corresponds to one of the 1-second time segments.

Thinking Stage 2

Imagine that the segment represents 100 meters and is cut up into a total of 9.81 equal segments. Each segment represents $\frac{1}{9.81}$ of the total distance.



100 meters

Each of these 9.81 distance segments is $\frac{1}{9.81}$ of 100 meters or $\frac{1}{9.81} \cdot 100$ meters or about 10.19 meters. Each of these 10.19-meter segments corresponds to 1 second, so we can say that Usain Bolt traveled, on average, 10.19 meters each second or 10.19 meters per second.

This reasoning creates the well-connected network of understanding that can lead to procedural fluency where students come to recognize the need to divide when computing an average rate of change. Furthermore, they can connect procedural and conceptual understanding while making sense of the average rate of change.

What is described above provides students with greater opportunity to develop procedural fluency as envisioned by the definition provided by NCTM at the beginning of this section.

In a second approach, students employ a more conventional routine where they compute Bolt's average speed by using the two points (0,0) and (9.81, 100). Using a formula which may only be memorized, students can compute as shown.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{100 - 0}{9.81 - 0} \approx 10.19.$$

The formula alone does not allow students the procedural flexibility and conceptual connections to make sense of the underlying mathematics and meaning of this calculation. However, once a conceptual foundation is established, students may be able to make sense of this traditional slope formula. Note that in the Usain Bolt context, $y_2 - y_1$ represents the total distance traveled and represents the elapsed time. Students could reason about the slope formula as follows:

- In $\frac{1}{x_2 - x_1}$ of the time, Bolt will run of the distance.
- That is, $\frac{1}{x_2 - x_1}$ of the elapsed time of $\frac{1}{x_2 - x_1}$ seconds represents 1 second.
- To preserve the proportional correspondence, Bolt will also travel $\frac{1}{x_2 - x_1}$ of the total distance or $\frac{1}{x_2 - x_1}$ $(y_2 - y_1)$ or $\frac{y_2 - y_1}{x_2 - x_1}$.

Furthermore, students who have learned mathematics with the intended balance between conceptual understanding, procedural fluency, and problem solving can extend their understanding of average rate of change to make sense of instantaneous rate of change, leading to the beginning development of the difference quotient and the limit definition of derivative.

CP. Conclusion

The learning of mathematical ideas is much more profound, learned information much more useful, and problem solving more deeply developed when ideas make sense, when students are active participants in the learning process, and when students have the opportunity for repeated reasoning. Much research demonstrates that when students simply memorize rules and procedures and perhaps try to remember using mnemonics, songs, and gimmicks, they struggle to recall the right procedure at the right time. When these procedures are not part of a well-connected web of understanding, they are not useful or not remembered correctly.

Students need to actively engage in the process of learning mathematical ideas, developing strong conceptual understanding, and using these ideas to develop procedural fluency. The traditional lecture format, where the instructor is engaged in mathematical thinking, will not accomplish the procedural fluency goals we desire. Consider the following from Freeman et al. (2014):

> In addition to providing evidence that active learning can improve undergraduate STEM education, the results reported here have important implications for future research. The studies we meta-analyzed represent the first-generation of work on undergraduate STEM education, where researchers contrasted a diverse array of active learning approaches and intensities with traditional lecturing. *Given our results, it is reasonable to raise concerns about the continued use of traditional lecturing as a control in future experiments* (emphasis added) (p. 8413).

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Assessment Practices

AP.1. Basics about assessment

The purpose of this chapter is to provide guiding principles for assessing students' learning of mathematics through summative and formative assessments. Instructors are likely aware of summative assessment whose purpose is to evaluate student proficiency with regard to one or more learning outcomes, such as exams, quizzes, and homework (when graded after only one attempt). Black and William (2009) define a formative assessment practice as one in which instructors elicit, interpret, and use evidence about student achievement to make decisions about the next steps in instruction with the goal of improving instruction and student learning. We elaborate on these assessments in section AP.1.3.

Assessment is regarded as an essential element for learning in terms of finding and recording increased knowledge or skills (Hanson and Mohn, 2011). Recently there has been an increase in this original purpose of assessment due to the rise of the accountability paradigm, including heightened inquiry into all aspects of the educational process by various entities, internal or external (Hanson and Mohn, 2011). We attempt to address both of these needs by providing various vignettes where summative or formative assessments are implemented. This introductory section gives an overview to allow independent reading of other sections based on what is most relevant to the reader.

Throughout the chapter we emphasize components of effective assessment which include

1. stating high-quality goals for student learning,
2. providing students frequent informal feedback about their progress toward these goals, and
3. evaluating student growth and proficiency based on these goals.

AP.1.1 Assessment frameworks

Several of the K–12 and collegiate professional organizations have developed frameworks for assessment. For example, in 1995, the National Council of Teachers of Mathematics (NCTM) published their third volume on standards, *Assessment Standards for School Mathematics*, which includes a useful framework for identifying the broad purposes and phases of assessment (NCTM, 1995, p. 27). The framework begins when mathematics instructors identify what they hope to accomplish in terms of student learning, instructional practices, student engagement, and motivation, all in the context of the broader mathematics program in which they live. This reflective assessment practice is cyclic, occurring in four phases as the instructor seeks to meet each specific assessment purpose: planning an assessment, gathering and interpreting data, and using evidence to modify future lessons.

At the collegiate level, the Mathematical Association of America (MAA) has published two volumes about assessment: *Assessment Practices in Undergraduate Mathematics* (Gold et al., 1999) and *Supporting Assessment in Undergraduate Mathematics* (Steen, 2006). Both of these volumes range in their scope, including assessment of students, teachers, programs, and majors. The American Statistical Association has published *Guidelines for Assessment and Instruction in Statistics Education, College Report* (GAISE, 2016), focused on assessment at the college level. The Society for Industrial and Applied Mathematics has published *Guidelines for Assessment and Instruction in Mathematical Modeling Education* (GAIMME, 2016), which is broadly applicable across courses and across student levels. All instructors in the mathematical and statistical sciences can benefit from consulting these resources as they develop assessment techniques for their courses.

Historically, the MAA has been a leader in encouraging thoughtful and ongoing assessment of student learning in undergraduate mathematics. In the introduction to the MAA's Notes volume *Assessment Practices in Undergraduate Education* (Gold, Keith, and Marion, 1999), Steen outlines six principles of assessment that remain relevant today. These principles have since been supported by additional research in mathematics education and they serve as useful background when integrating assessment information into practice (Soto-Johnson and Fuller, 2012; Soto-Johnson, Yestness, and Dalton, 2009). Steen recommends that assessments (1) be on continuous cycle, (2) serve as an open process, (3) promote valid inferences, (4) employ multiple measures of performance, (5) measure what is worth assessing, and (6) support every student's opportunity to learn important mathematics. These principles mirror the NCTM's (1995) recommendations regarding assessment. The NCTM stresses how assessments should reflect the mathematics that students should know, enhance mathematics learning, promote equity and valid inferences, and be an open and coherent process. These recommendations suggest that assessments should inform instructors about their students' learning as well as about their teaching. Below we elaborate on each of Steen's principles of assessment and briefly discuss how we connect to them in this chapter.

Principle 1. Assessment is not a single event but a continuous cycle.

Many of the vignettes in this chapter focus on a particular problem facing a teacher, but the instructor, the department, and indeed the whole institution should develop a culture in which assessment is ongoing.

Principle 2. Assessment must be an open process.

Approaches to assessment should inform both teaching and learning—that is, it informs both instructors and students. Open communication about learning goals as described in section AP.1.2 is important. In practice for instructors, this means that we use assessment to inform students of their progress and not just as a grading tool.

Principle 3. Assessment must promote valid inferences.

Results of good assessments promote responses that directly impact classroom instruction. We try to frame these approaches based on the goals of particular courses.

Principle 4. Assessment that matters should always employ multiple measures of performance.

Historically the mathematics community has based summative assessment on high-stakes tests in a few standard formats. We present numerous ways to encourage a wider set of measures. In practice this means that instructors use assessment to inform students of their progress.

Principle 5. Assessment should measure what is worth learning, not just what is easy to measure.

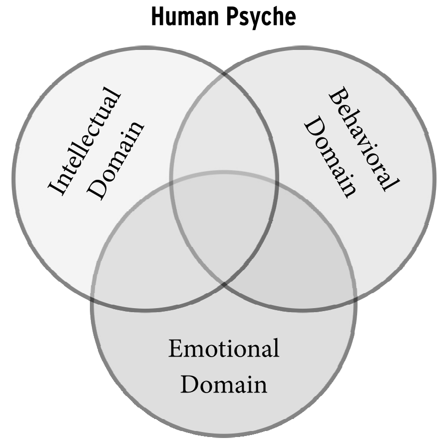
The balance between the depth of what we can measure for each student and the practical limitations imposed by real world constraints is highlighted in our sections on large classes and online learning.

Principle 6. Assessment should support every student's opportunity to learn important mathematics.

Good assessment practices are vital in providing opportunities for a larger and more diverse set of students to participate in mathematics.

AP.1.2 Clearly specify learning outcomes

Mathematicians often use the phrase "mathematical maturity" to capture the idea that a student is mathematically well-rounded, not only in mathematical content knowledge but in other ways as well. A useful oversimplification frames the human psyche as a three-domain model. The *content domain* or *intellectual domain* regards knowledge and understanding of concepts. The *cognitive domain* or *behavioral domain* regards the practices and actions with which we apply or develop that knowledge. The *affective domain* or *emotional domain* regards how we feel about our knowledge and our actions. All three domains play key roles in student learning and contribute to developing students' mathematical maturity. As such, in our mathematics courses it is important to create learning outcomes that include goals representing these three domains.



Course learning outcomes will vary across courses and institutions, but they should have in common this broad representation of areas of learning that extend beyond a list of content topics. Learning outcomes for students provide the definition of student success in a course, and all decisions made by instructors regarding assessment, in conjunction with classroom practices, should be directly aligned with these goals.

Careful, objective wording is often encouraged when phrasing goals for the purpose of assessment. For example, "understand" is often considered too vague to capture an assessable objective. Consider the statement, "Students will understand the fundamental theorem of calculus." This student learning goal is about a specific concept within a calculus course, but does not provide a measurable outcome: it is not clear how "understand" is to be assessed, or what particular instructional design may be best for developing such understanding. By contrast, consider the statement, "Students will explain the significance of the fundamental theorem of calculus and use the theorem appropriately." This second statement provides two measurable outcomes: Students can explain the significance of the fundamental theorem of calculus and students can apply the fundamental theorem of calculus appropriately. The phrase, "explain the significance" demonstrates knowledge of the concept and why it is important within the context of calculus, while the phrase, "can apply" demonstrates skill in recognizing the applicability of the theorem as well as ability to use the theorem.

Below we provide a vignette that exemplifies the three domains.

Vignette: Professor Johnson is teaching a number theory class and a college algebra class. In the number theory class, Professor Johnson wants to make sure that students not only learn definitions, theorems, and proof-writing, but also that students learn how to use computer algebra systems to conduct experiments with modular arithmetic. In the college algebra class, Professor Johnson wants students to learn mathematical procedures and develop an understanding of how they work. After reviewing the *Common Core Standards for Mathematical Practice* (National Governors Association Center for Best Practices, 2010) the *CUPM Guide to Majors in the Mathematical Sciences* overview (MAA, 2015), and the *MAA CRAFTY College Algebra Guidelines* (MAA, 2007), Professor Johnson created sets of learning outcomes to include in the two syllabi, a few of which are listed below.

Number Theory

Students will

- Explain the connections between divisibility, the division algorithm, and the Euclidean algorithm.
- Use Fermat's Little Theorem and Wilson's Theorem to solve problems involving linear congruences.
- Create proofs using a variety of techniques, including direct proof, proof by contradiction, and mathematical induction.
- Experiment with various computational approaches.

Students will have opportunities to engage in the following general behaviors and mindsets—i.e., **mathematical practices**:

- Persist, work through perceived failure, and engage in strategic self-questioning.
- Collaborate productively with others and ask good questions.
- Construct examples and counterexamples to investigate and understand new definitions and theorems.
- Read and understand existing proofs and recognize incorrect proofs.
- Create and communicate original proofs.

College Algebra

Students will

- Use multiple perspectives (symbolic, numeric, graphic, and verbal) to explore elementary functions.
- Algebraically solve linear, quadratic, exponential, logarithmic, and power equations.
- Sketch polynomial and rational functions using a graphing calculator.
- Identify and algebraically find important characteristics of these graphs such as intercepts, vertical asymptotes, and horizontal asymptotes.
- Recognize and use standard transformations with graphs of elementary functions.
- Use and solve systems of equations to model real world situations.

Students will have opportunities to engage in the following mathematical practices:

- Persist and work through perceived failure.
- Collaborate productively with a team.
- Develop a personal framework of problem solving techniques (e.g., to make sense of problems, sketch and label diagrams, restate and clarify questions, identify variables and parameters, and use analytical, numerical, and graphical solution methods).
- Create, interpret, and revise real-world models and solutions of problems.

Discussion: Professor Johnson divided the learning outcomes for each class into mathematical content topics, which primarily represent the intellectual domain, and mathematical practices, which represent the cognitive and affective domains. By relying on existing recommendations from professional societies and other reports, Professor Johnson was able to identify examples from which to obtain inspiration and to adopt language.

Practical tips

- Create learning outcomes based on course, departmental, and personal objectives.
- Generate ideas of learning outcomes that fit with the behavioral and emotional domains. See the *MAA Curriculum Guide* and the *Common Core Standards* along with the "five-strand" model discussed in the report *Adding It Up* (National Research Council, 2001).
- Document learning outcomes in writing to develop your course or to include in your syllabus for students.

When creating learning goals there is a natural tension between Steen's third principle which recommends creating valid inferences and measurable learning goals and the fifth principle that suggests that some key elements of understanding are inherently difficult to measure. Professor Johnson's goals for number theory seem more in line with the fifth principle, while it is easier to imagine measurable results for the college algebra course. Finding the balance between these principles is a challenge but is also a key to good assessment. The Design Practices chapter contains an extensive discussion on creating student learning outcomes.

AP.1.3. Formative and summative assessment

Assessment can be roughly divided into two types, formative and summative, but these need not occur in isolation. In fact many standard assessment techniques have elements of both. As discussed in the introduction formative assessments are generally incorporated to elicit, interpret, and make decisions about the next steps in instruction. When we want to provide feedback to students, before we actually evaluate their work, we need to conduct some informal assessment—this is always formative in nature. While techniques such as observing students work in groups or having students answer verbal questions during class are well-known formative assessments, there are other surprising techniques in which formative assessment can arise. For example, online homework systems can be set to allow students multiple attempts on each problem without penalty. This can provide students with feedback on the correctness of their work and provide instructors with information about which problems are most difficult for students, and thus, affect an instructor's subsequent teaching.

While formative assessments are primarily used to inform the direction in which instructors might modify their lessons, summative assessments are conducted with the purpose of evaluating student proficiency with regard to one or more learning outcomes. However, looking over the learning outcomes from our earlier vignette with Professor Johnson, it is apparent there are many learning outcomes for which exams, quizzes, and homework are not effective summative evaluations. Sometimes these outcomes are simply difficult to evaluate. In such cases assessments such as writing assignments, group projects, and oral presentations are appropriate summative evaluations, but they can also have a formative component when coupled with appropriate feedback. Many assessment tools have both summative and formative components.

In discussions regarding the use of formative and summative assessment, a distinction is often made between assessing growth versus assessing proficiency. In measuring student **growth** we evaluate students' progress compared to their starting point. In measuring student **proficiency** we evaluate students against a fixed target outcome, regardless of their starting point. In K–12 educational policy, decisions about the relative importance of assessing growth versus proficiency have major impacts in terms of how schools are evaluated. At the postsecondary level, it is important to determine whether growth or proficiency is the primary goal when setting learning goals and selecting formative and summative assessments. In STEM courses such as calculus, linear algebra, and differential equations, it is reasonable for proficiency to be the primary goal. In non-STEM courses, such as those that satisfy general education or quantitative literacy requirements, it is equally reasonable for growth to be used as the primary focus of assessment. In other courses, growth and proficiency might be given equal weight when assessment methods are chosen. Regardless of how this distinction is handled in a given course, the important point is that one should clearly decide and articulate how to balance the assessment of growth and proficiency.

Following are independent sections on formative and summative assessment, but because these assessments inform one another there is some overlap in the sections. Feedback is an essential part of both formative and summative assessment as reflected in Steen's principles #1 (continuous cycle), #2 (openness), and #6 (enabling students to connect to mathematics). Instructors need to be aware that feedback is constant through classroom behaviors as well as through graded assignments or classrooms discussions.

AP.2. Formative assessment creates an assessment cycle

How do students obtain informative feedback about their understanding of a given topic or about the effectiveness of their approach to solving problems? How do instructors determine how students "are really reasoning" about mathematics? How do we ensure students and instructors have the same understanding of the course goals? The answers to these and similar questions fall within the scope of techniques of formative assessment.

AP.2.1. Implementing formative assessment

Black and Wiliam (1998, 2009) identify five strategies for implementing formative assessments.

1. Clarify and share learning intentions and criteria for success.
2. Engineer effective classroom discussions and other learning tasks that elicit evidence of student understanding.
3. Provide feedback that moves learners forward.
4. Activate students as instructional resources for one another.
5. Activate students as the owners of their own learning.

Each strategy is an action for an instructor to implement. However, the degree to which one implements each strategy, and when, are questions to consider.

- Does the instructor tweak a problem in a subsequent assignment, or adjust activities and actions for the entire next class period?
- Is an instructor always eliciting feedback and adjusting their teaching on the spot, or is the instructor adjusting subsequent class periods using feedback?
- How does the instructor utilize summative assessments in conjunction with formative assessments?
- Does the instructor adjust their course after the semester or during the course?

Although these are rich suggestions and questions to ponder as one implements formative assessments, these are also challenges to consider. Heritage et al. (2008, p. 24) finds that "teachers do better at drawing reasonable inferences of student levels of understanding from assessment evidence, while having difficulties in deciding next instructional steps." Millar (2013, p. 55) states that "formative assessment is more effective when it avoids comparing one learner with another, or with a group norm, but instead focuses on providing task-related feedback that helps the learner to better understand the relationship between an aspect of their current performance and the desired performance." Therefore, the quality of formative feedback is essential: "The premise underlying most of the research conducted … is that good feedback can significantly improve learning processes and outcomes, if delivered correctly" (Shute, 2008, p. 154).

Formative assessment can occur even after a formal quiz or exam. For example, instructors might write questions for subsequent quizzes or homework that will adjust for the lack of student understanding of particular content on a previous exam. Flexibility with a variety of formative assessment techniques, and allowing students to find their own paths towards understanding the material, seems both to increase performance by students marginalized by lecture-based teaching (Laursen, Hassi, Kogan, and Weston, 2014) and to cultivate success in subsequent courses because the students are empowered to learn (Hassi and Laursen, 2015).

There has, of course, much research has been conducted on formative assessment in undergraduate mathematics education (e.g., Reinholz, 2016), and guides about undergraduate mathematics assessment have been published (e.g., Gold, Keith, and Marion, 1999). In recent years, technology-enabled formative assessment has become an integral part of many undergraduate mathematics departments. Adaptive computer systems such as **A**ssessment and **Le**arning in **K**nowledge **S**paces (ALEKS, McGraw-Hill Education), which is used as a placement mechanism in some higher education institutions, utilize "adaptive questioning" by periodically reassessing what content a student knows about a particular mathematical topic (Sullins et al., 2013). In other words, ALEKS utilizes student answers to pose subsequent questions, thereby using previous answers in a formative way.

Discussion on the delivery of formative assessment and feedback accompanies each of the three vignettes below.

Vignette 1

An instructor, Dr. Doe, is not sure her students understand concepts already covered in her precalculus and trigonometry course. During the next class period she gives a quiz that includes questions related to those concepts. After grading the quiz, she concludes that students did not perform well on questions related to trigonometric identities. Thus, she spends the first twenty minutes of her subsequent class to engage students on identities before moving on to a new concept.

Discussion: The assessment, a quiz, was formative in that Dr. Doe adjusted her next class period to address student difficulties by focusing more time on identities. Dr. Doe also performed this adjustment for students' growth, knowing that perhaps she could influence their understanding of the topic. Dr. Doe exhibited two of the five key strategies of Black and Wiliam (2009), eliciting evidence of student understanding from a learning task using the quiz and providing feedback that moves learners forward by explicitly addressing the difficulty. However, one could question the depth of both the evidence of student understanding and the action taken to provide feedback. In this vignette no detail is provided regarding the rest of the class periods. Dr. Doe could have purposefully assigned more trigonometric identity problems in future tasks or exams. If so, this would be an example of interleaving, where tasks are spaced and repeated during the course with the goal of retaining the concept longer (Roediger and Pyc, 2012).

Vignette 2

Dr. Smith grades the first test in her course. She decides that the students need to reflect on their own performance on the exam. Therefore, the students are given an optional assignment in which they redo their test, correcting any mistakes, and write a one-page paper about their problem-solving process, including what they must do to improve their performance for the next test. Dr. Smith attaches a rubric detailing how she will grade the one-page paper.

Discussion: This example shows how a traditionally summative assessment, an exam, can be used as a formative assessment. Dr. Smith took an exam and parlayed it into a potential learning experience for the students. She achieved three key strategies by sharing criteria for success (by providing a rubric), eliciting evidence of student learning (by grading the assignment), and activating students as owners of their learning (by asking them to reflect on their mistakes and how to improve). The last strategy of the assignment is a meta-cognitive approach—that is, instead of focusing on the problems and solutions, the students are asked to improve their own problem-solving process.

Vignette 3

Dr. Blue teaches calculus using pedagogical student-centered techniques. On this particular day, students come into class, form into pre-determined groups of three, and work on definite integral problems for fifteen minutes. After the fifteen minutes, one group demonstrates a solution to the rest of the class. Dr. Blue asks the class if they have any questions, and one student poses a general (and unknown to the students) conjecture about integrals to infinity. At a pedagogical crossroads, Dr. Blue chooses to field the question by allowing the groups to discuss their answers to the student's question for five minutes. After this another group shares their answers, and Dr. Blue ends the class with her thoughts about improper integrals.

Discussion: Improvisational teaching may occur when students pose questions or thoughts that are unanticipated by the instructor. This scenario has a student who is mathematically curious about a certain topic, which can be beneficial to both the student and the class (Knuth, 2002). In a student-centered classroom where the instructor is regularly soliciting student reasoning, formative assessment is used almost constantly. Dr. Blue used two key strategies. First, she engineered classroom discussions to elicit evidence of student understanding by giving them time to work on the definite integrals. Second, she activated students as instructional resources for one another by asking students to present their solutions and also by giving them time to think about and answer the student's question. The instructor could have answered the question by either deferring it to the respective section in the book or answering it herself. However, Dr. Blue chose to take class time to allow the students in groups to conjecture answers, thus passing the ownership of knowledge to them. The pedagogical choice she made after the assessment task could be considered the key strategy, activating students as the owners of their own learning.

Summary discussion

Each of the three vignettes utilizes formative assessment and feedback in different depths and with different foci. In the first vignette, Dr. Doe's formative feedback focuses the class again on trigonometric identities. While this may indicate that the instructor is adjusting her teaching and valuing students' understanding, another discussion or lecture may not be as deep in student learning as having students engage in trigonometric identity problems in class or create their own trigonometric identity problems to present in class. The second vignette focuses on problem solving instead of topics, and the third focuses on ownership of generating mathematics. In all cases, one item is universal: the instructor adjusts her teaching in an effort to improve her students' understanding of mathematics.

One aspect to consider when thinking about formative assessment is the negation: what does a classroom without formative assessment look like? The instructor would have a predetermined plan, both with daily activities and assessments and would never adjust that plan. The instructor would not alter future courses with the knowledge and feedback from the previous course. While this extreme case may not be common, there is no doubt that the amount of adjustment, the depth of assessment, and the level of feedback all vary greatly between instructors.

Therefore, it seems that a prevailing philosophical notion underlies formative assessment. The teacher's belief of what mathematics is, and how it should be presented or taught, is visible from each action in the classroom. Formative assessment may reflect that mathematics can be approached with a growth mindset (Dweck, 2008) instead of fixed—that is, students can be arbiters of their own study of mathematics instead of "never being able" to learn it. In vignette 2, Dr. Smith values the problem-solving process, perhaps because she philosophically believes that mathematics is more about process than product. In vignette 3, Dr. Blue allows conjecturing to take place in the classroom, valuing unknown situations over known quantities.

Finally, formative assessment will be different depending on the environment in the classroom. However, some of the key strategies by Black and Wiliam (2009) may not work as well in a traditional lecture-based course. For example, "engineering effective classroom discussion to gain evidence of student understanding" and "activating students as instructional resources for one another" may happen more naturally in a student-centered course, where most of the time students are drivers of content during class. Formative assessment can only be as effective as the instructor values it and as the students embrace it.

Practical tips

- Formative assessment is about soliciting evidence to adjust teaching with the intention that student learning improves.
- Summative assessments (as discussed in the next section) can be made formative and may be beneficial for students to value reflection.
- Think about the formative feedback that you communicate. This reflects your beliefs about what is important in mathematics.
- Prompt, specific feedback on students' strengths and weaknesses is crucial to helping students understand how they can improve.
- Create formative assessments that align with Steen's principles.

AP.2.2. Formative assessments to improve mathematical practices

How can formative assessment be used when trying to understand the emotional and behavioral components of student learning? The general behaviors that students bring to mathematical work are often referred to as their **mathematical practices** (CCSSM, 2014). We next discuss an example of how formative assessment can be used to influence and develop these practices.

Vignette

To assist students in developing the habit of making sense of problems, i.e., of "fiddling" with a problem to gain insight, students are introduced to a game from Vandervelde (2010) called *Last One Standing*. Here are the rules, as played with four people:

1. The players are all sitting in a row, oriented so that one person is in the back and another is in the front. The person in the rear can see the others, while the person in front cannot see anyone else.
2. All individuals should be seated initially.
3. The game ends when the last person in line is standing while all others are seated.
4. The person in front can stand or sit at will.
5. The other three participants can move according to the following rule: a person may change state (by standing up or sitting down) if the person immediately in front of them is standing, while all others in front of them are seated. Otherwise they are locked in position and may not move. The status of people behind them does not matter.

Students are asked to gather in groups and play this game several times to ensure they understand the rules. Two volunteer groups play the game to ensure the whole class agrees on the fewest steps it takes to complete the game with four people.

The class is then told (possibly during the following class period) the fewest steps for a game with 20 people playing and asked to discuss the fewest steps for a game with 21 people playing. The students are posed the question, "What questions need to be asked and answered in order to tackle this problem?" This creates strategic confusion among the students, leading to productive discussion and an opportunity for formative feedback.

Discussion: In practice, students are generally engaged in trying to understand the growth pattern of the number of moves. Some groups become convinced it should be linear and are puzzled when it doesn't work with the small numbers that they try, going back to two people, then three, then five. Some groups see growth patterns they are not able to articulate algebraically, while other groups attempt to write equations in recursive or closed forms.

The goal of this task is not necessarily to solve the problem, but rather to help students develop good mathematical habits of making sense of problems and being persistent. For students who have previously engaged in activities only superficially, the task provides a new opportunity for them to dive in and try to identify mathematical patterns, allowing greater freedom and creativity in the problem-solving process. An important caveat is that the feedback loop from this sort of assignment must take into account that some students, particularly those facing stereotype threat or language barriers, may feel intimidated by this non-traditional activity. As Steen suggests, ensuring assessment is cyclic is vital to evaluating the effectiveness of these activities.

The activity situates the class to appreciate the value of playing with examples in an effort to solve a problem. It also helps students move from recognizing a numerical pattern to expressing it algebraically, and finally to discussing the value of different representations.

Practical tips

- When providing feedback to students, remain process-oriented and question-based:
  - What is your motivation for trying that?
  - Have you tried working through the simplest cases?
  - What does the rest of your group think about this idea?
- Focusing feedback on the mathematical process keeps the emphasis on mathematical behaviors and away from students' emotional responses to challenging problems.

Interactions with students regarding mathematical practices do not need to be as formal as the previous vignette suggests. There are many unplanned thoughts by students shared during class time, some of which lead to explicit demonstrations of better mathematical practices. For example, when discussing in first-year calculus whether a series converges or diverges, a student might ask the question, "Can you have a series of entries that are themselves series?" At that moment instructors have several options. They may choose to directly answer the question with a mathematically correct response, or they may tell the student that this is not covered in the material for this course, but they will see it if they take more courses. A third option is to change the focus of the class from the topic of series to a holistic discussion of mathematical practice, pointing out how this student's question is a great example of mathematical thought. This illustrates Black and Wiliam's (2009) first strategy.

AP.2.3 Formative assessment to influence students' beliefs and motivations

Formative assessment can also be used to influence student beliefs and motivation, as the following vignette suggests.

Vignette

On the first day of a mathematical content course for elementary education majors, Professor Somerville asks the following questions: What is this class about? What more is there to learn? Don't you already know all the mathematics that you will be teaching in elementary school? Aren't you going to take a methods course anyway? These questions appear to suggest that not much mathematics is required to teach elementary students. They also imply that computational proficiency and general teaching skills are sufficient for good mathematics instruction in elementary school. Professor Somerville then defines mathematical knowledge for teaching (Ball, Hill, and Bass, 2005) and launches an activity.

The students are asked to compute 35×25 by themselves and to compare answers with their neighbor. They are asked to compare how they solved the problem. Generally everyone uses the standard algorithm, and they obtain the correct answer.

The instructor explains that the ability to carry out this computation is an example of a basic skill that they possess. Then they are shown the slide reproduced in Figure 1.

They are asked to identify the method each student used, to determine why the method works, and to determine if each student's method will always work. Most students, after intense group work, can identify the first two students' reasoning. Groups are asked to volunteer their explanations and the class discusses their comments. The last student's work eludes all but one or two students, who still have a difficult time articulating the actual thought process and explaining it to their peers.

In the wrap-up after the class discussion, Professor Somerville explains to the students that possessing basic skills allows them to determine if a final answer is correct, but mathematical content knowledge allows them to answer the type of questions posed with this activity.

Student A	Student B	Student C
35	35	35
× 25	× 25	× 25
125	175	25
+75	+700	150
875	875	100
		+ 600
		875

Figure 1. Computations using three methods (Hill, Rowan, Ball 2005).

Discussion: The purpose of this task is to

- Change the students' mindsets about the usefulness of the course, which is often regarded as covering content that is too in-depth.
- Explain to students the difference between knowing mathematics and knowing the particular mathematical content knowledge necessary for teaching.

In general, this activity unsettles the students, who often express concern about their ability to perform this analysis in real time. The final discussion allows the instructor to frame the purpose of the course, and to establish a reference point adopted throughout the course which is to distinguish between "How to …" and "Why …".

AP.3. Summative assessment

As discussed in the first section of this chapter, summative assessment is conducted with the purpose of evaluating student proficiency with regard to one or more measurable learning outcomes. In this section we discuss various ways to structure course assessments and effective methods for designing summative assessments.

AP.3.1 Assigning course grades

Traditionally, mathematics grades are determined by using a points-based system—that is, by assigning points to each assessment item and computing an average to produce a course grade. This is a reasonable approach as long as the average is created in a thoughtful manner that reflects the student learning outcomes for the course.

Vignette

Professor Jackson identified nine student learning outcomes for a first-semester course primarily designed to serve future engineers. Six of these learning outcomes were content-based outcomes and three related to student mathematical practices. In order to emphasize the seriousness of the practice outcomes, Professor Jackson realized that assessments needed to extend beyond problem sets and exams. To align the overall course grade scheme with the student learning outcomes, Professor Jackson structured the weights for each assessment item as follows:

Three Exams: 45% of course grade

Quizzes: 15% of course grade

Written Assignments: 5% of course grade

Online Homework (repeated attempts allowed on problems): 15% of course grade

Three Reflective Essays: 15% of course grade
Participation: 5% of course grade

Discussion: Professor Jackson decided to allocate 65% of the course grade to summative assessment via exams, quizzes, and written assignments, 15% of the course grade to formative assessment via online homework, and 20% of the course grade to participation and essays that reflect the practice outcomes. Because the first semester of calculus establishes a foundation for multiple semesters of study, Professor Jackson felt it was important to have summative assessments encompass a significant portion of the course grade. Professor Jackson felt it was also important to have a non-trivial percentage of the grade reflect persistence, effort, and engagement in the course. By distributing grades across a range of assignments that were aligned with the course learning outcomes, Professor Jackson was able to clearly communicate to students through the grading scheme that all of these outcomes are critical for mathematical learning.

While points-based systems are reasonable, using other course grading schemes can positively impact student attitudes and perceptions about their learning. One example of this is the student course portfolio, which is a common means of assessment in humanities and art courses. Portfolios can serve as the single source of a course grade, given that they represent the work of a student throughout a course, but they can also serve as one component of a course grade. While there are some documented examples of the use of portfolios for undergraduate mathematics courses (Burks, 2010; MAA, 1999), the use of portfolio grading has been more common in K–12 mathematics courses than at the postsecondary level.

Vignette

In a junior-level course on mathematical problem solving that primarily serves preservice teachers, Professor Klein assigns the final course grade with 10% based on attendance and participation and 90% on student portfolios. The professor collects the portfolios four times during the semester and offers opportunities for students to revise and resubmit quizzes and problem sets. The grading scheme for the portfolio is described in the syllabus as follows.

1. **Journal of strategies and methods**: Students are expected to maintain a list of strategies and mathematical methods (e.g., induction, contradiction, etc.) that are discussed during class. This is meant to be a typed encyclopedia, without any reflective component. The journal portion of the portfolio is worth 5% of the portfolio grade, and is assessed based on completeness.
2. **Quizzes**: Quizzes take place approximately once per week in this course and are announced in advance. Each quiz consists of one exercise or problem and lasts 25 minutes. After Professor Klein grades and returns all quizzes the students add them to their course portfolios. There are a limited number of revision opportunities for quizzes. The total quiz score is worth 35% of the portfolio grade.
3. **Problem Solutions**: The problems section of the portfolio has two subsections, one for solutions to problems discussed during in-class small group work and one for solutions to homework problems. Homework problems are graded after initial submission. Select homework problems are eligible for revision and re-grading as a part of the portfolio. Each problem solution is contained on a separate page in the portfolio. The problem score is worth 40% of the portfolio grade.
4. **Reflections or cover letters**: Each time the portfolio is collected, students include a cover letter detailing their reflections on their work to date, their progress toward the student learning outcomes, and specific items in the portfolio that illustrate the key points of their reflection. The reflective essay score is worth 20% of the portfolio grade.

Discussion: The two key differences between Professor Klein's portfolio-based grading system and many traditional points-based systems are (a) students have opportunities to revise and resubmit selected quizzes and problem sets, and (b) students are expected to collect and review their work periodically throughout the course and write cover letters reflecting on their learning. By focusing students' attention on their work through the lens of the course portfolio, Professor Klein can inspire a more holistic perspective on growth and achievement compared to other assessments.

Other alternatives to traditional points-based systems and portfolios can be found in variants of **mastery-based grading** such as **standards-based grading** and **specifications grading**. In a generic mastery-based grading system, an extensive itemized list of learning outcomes for the course is provided. Students are then expected to demonstrate mastery of each individual learning outcome, and the number of learning outcomes for which mastery has been achieved determines the course grade. There are subtle and influential differences across various implementations of this type of grading system. A comprehensive guide to implementing mastery-based systems is beyond the scope of this guide, but there are many resources that serve this purpose. For example, Pengelley (2017) describes how one can incorporate entirely qualitative rubric-based grading resulting in only letter grades rather than point-based grades for all assignments. Nilson (2014) provides a comprehensive guide for instructors regarding specifications grading though it is not specific to mathematics. The use of standards-based grading in postsecondary mathematics courses has been discussed in a variety of articles and blog posts (e.g., Brilleslyper et al., 2011; Owens, 2015).

Practical tips

- When constructing a course grading system, be mindful of constraints such as available instructor time, number of students, and availability of teaching assistants. It is usually better to implement a lower-impact grading system with high quality than to poorly implement a more ambitious grading system.
- It is critical that students understand how the grading system for their course works. For courses that incorporate specifications or standards-based grading, a significant amount of time and energy must be spent early on getting students to understand and buy into the system.
- For courses that use a portfolio as a large component of the overall grade, it is critical that the portfolio be collected multiple times during the semester so that students understand how the portfolio is assessed. This also supports Steen's assessment as a continuous cycle.

AP.3.2. Exemplary summative assessments

As previously mentioned summative assessments are evaluations of student mastery of topics directly related to student learning outcomes, occurring at the end of an instructional unit. These can include evaluations of procedural fluency, of conceptual understanding—a topic of discussion in a separate section of this chapter as well as in the Classroom Practices chapter, of written or oral communication in mathematics, of problem-solving strategies, or of other elements of mathematical proficiency. Familiar examples of summative assessments include exams, performance tasks, projects, and portfolios. There are several aspects of effective summative assessments that are not immediately apparent, as we shall discuss in this section.

A key element of summative assessment often overlooked is the effect of prior mathematical experiences on student learning. To account for this effect, it is helpful to conduct summative assessments prior to teaching. This pre-assessment of student knowledge should not be counted toward student course grades, making these pre-assessments formative in some sense. However, the purpose of these early assessments is specifically to evaluate student learning, since at the beginning of any unit of study certain students are likely to have already learned some of the skills that the instructor is about to introduce, others may already understand key concepts, and others still may be deficient in prerequisite skills or have misconceptions. Equipped with diagnostic information from pre-assessments, an instructor gains greater insight into what to teach by knowing what skill gaps to fill, by initiating activities based on preferred learning styles, and by connecting the content to students' interests. Teachers can use a variety of practical pre-assessment strategies, including pre-tests of content knowledge, skills checks, and concept maps. In addition, powerful pre-assessments have the potential to address a worrisome phenomenon reported in a growing body of literature (Bransford, Brown, and Cocking, 1999; Gardner, 1991): a sizeable number of students come into mathematics courses with misconceptions about both subject matter and themselves as learners. If teachers don't identify and confront them, the misconceptions will persist even in the face of good teaching.

Assuming that effective pre-assessments and formative assessments have been implemented, our next goal is to describe characteristics of exemplary summative assessments. Exemplary assessments, whether classified as formative or summative, are meaningful, motivational, engaging, and should guide the student in the learning process (Huba and Freed, 1999; Walvoord and Anderson, 1998), but most importantly they should be in line with the course learning outcomes. Huba and Freed identified eight characteristics of exemplary assessments, and many of these reflect Steen's principles of assessment. The characteristics are

- Authentic — reflect real life experiences
- Challenging — stimulates the learner to apply knowledge
- Coherent — serves as a guide for the student to achieve the learning goal
- Engaging — attracts the learner's interest
- Respectful — sensitive to the individual learner's beliefs and values
- Responsive — includes a feedback mechanism to assist the student in the learning process
- Rigorous — requires applied understanding of learning to achieve a successful outcome
- Valid — provides information that is useful to meet the intended learning outcomes

The vignette below exemplifies the use of exemplary assessments, but the reader should recognize that an assessment can be exemplary without satisfying all of the above criteria.

Vignette

In an online college algebra course, Professor Agnesi has exclusively used multiple-choice questions on computer-based exams taken at multiple, proctored environments physically separated from each other. The professor is concerned that while the exam problems are rigorous and challenging, the exams are not in overall alignment with the learning outcomes for the students. With Steen's fourth principle (multiple forms of assessment) in mind, and in an effort to increase the coherence of the exams with the learning outcomes, to increase the level of engagement of the problems, and to increase the level of feedback that students can attain, Professor Agnesi decides to implement the following guidelines for all exams in the course.

1. Create questions in multiple formats so that guessing is minimized.
2. Use an online homework system or dedicated exam software to administer exams, with the capability of automated answer checking for problems that are not multiple choice.
3. Randomize the numerical values in the problems given to students.
4. Allow two attempts on each problem where students enter either a function or number as their answer.
5. Insert at least one higher-order thinking question in short answer or essay form.

Discussion: By considering each aspect of exemplary assessment, Professor Agnesi was able to implement changes that increased the quality of summative assessments given to the students. While some of the qualities of exemplary assessments were not addressed, these can be returned to at a later time after current changes have been streamlined and effectively implemented. Professor Agnesi made a conscious choice to focus on only a small number of these items for each round of modifications to summative assessment structures, with a long-term plan for making additional changes.

AP.3.3. Creating and selecting problems for summative assessment

The creation and selection of problems effective for summative assessment is a challenge that shares many qualities with designing and selecting appropriate mathematical tasks for students in the classroom. It is therefore recommended that this section be read in parallel with the section in the Classroom Practices chapter on selecting appropriate mathematical tasks.

The key to creating and selecting problems for summative assessment is to have a clear sense of what the problem is intended to assess and what it actually assesses. While it is not possible to know what a problem truly assesses, it is possible for teachers to evaluate tasks informally using review frameworks to check that the intended assessments are reasonably aligned with the assessment items. A well-known framework for analyzing problems is Bloom's taxonomy and its variants (Anderson et al., 2001; Bloom et al., 1956). Bloom's original work outlines multiple levels of skills in the cognitive domain of learning, increasing from simple to complex. These are described by six skill levels: knowledge, comprehension, application, analysis, synthesis, and evaluation. This work has been extended by researchers in educational psychology to more robust frameworks. For example, Anderson et al. (2001) introduces a two-dimensional extension of Bloom's taxonomy, pictured in the table below. The first dimension consists of a cognitive process dimension (remember, understand, apply, analyze, evaluate, create) similar to Bloom's taxonomy, while the second dimension consists of a knowledge dimension (factual knowledge, conceptual knowledge, procedural knowledge, and metacognitive knowledge). When evaluating a task using this taxonomy, the cognitive process is represented by the verb used in specifying the task (what the student is doing) and the knowledge process dimension corresponds to the noun (what kind of knowledge the student is working with). Examples of this extended taxonomy can be found in a special issue of the journal *Theory Into Practice* (Anderson, 2002), where specific applications to assessment issues are discussed (Airasian and Miranda, 2002).

		Cognitive Process Dimension					
		Remember	Understand	Apply	Analyze	Evaluate	Create
Knowledge Dimension	Factual						
	Conceptual						
	Procedural						
	Metacognitive						

Knowledge Dimension

1. **Factual knowledge:** The basic elements that students must know to be acquainted with a discipline or solve problems in it

a) Knowledge of terminology

b) Knowledge of specific details and elements

2. **Conceptual knowledge:** The interrelationships among the basic elements within a larger structure that enable them to function together

a) Knowledge of classifications and categories

b) Knowledge of principles and generalizations

c) Knowledge of theories, models, and structures

3. **Procedural knowledge:** How to do something; methods of inquiry, and criteria for using skills, algorithms, techniques, and methods
   a) Knowledge of subject-specific skills and algorithms
   b) Knowledge of subject-specific techniques and methods
   c) Knowledge of criteria for determining when to use appropriate procedures
4. **Metacognitive knowledge**: Knowledge of cognition in general as well as awareness and knowledge of one's own cognition
   a) Strategic knowledge
   b) Knowledge about cognitive tasks, including appropriate contextual and conditional knowledge
   c) Self-knowledge

Cognitive Process Dimension

1. **Remember**: Retrieving relevant knowledge from long-term memory
   a) Recognizing
   b) Recalling
2. **Understand**: Determining the meaning of instructional messages, including oral, written, and graphic communication
   a) Interpreting
   b) Exemplifying
   c) Classifying
   d) Summarizing
   e) Inferring
   f) Comparing
   g) Explaining
3. **Apply**: Carrying out or using a procedure in a given situation
   a) Executing
   b) Implementing
4. **Analyze**: Breaking material into its constituent parts and detecting how the parts relate to one another and to an overall structure or purpose
   a) Differentiating
   b) Organizing
   c) Attributing
5. **Evaluate**: Making judgments based on criteria and standards
   a) Checking
   b) Critiquing
6. **Create**: Putting elements together to form a novel, coherent whole or make an original product
   a) Generating
   b) Planning
   c) Producing

One productive way to envision robust summative assessment is as a collection of tasks that evaluate students across multiple levels of both the cognitive process and knowledge dimensions of this taxonomy. When creating exams, quizzes, or other assessments, instructors should intentionally create or select tasks that cover a broad spectrum of this framework. Alternatively, after instructors have created an assessment, it can be reviewed to check that it is broadly evaluating students across multiple dimensions. Guided by the descriptions of the framework components, the table above can be used as a primary ingredient of assessment design and evaluation[1], as demonstrated in the following vignettes.

Vignette 1

Professor Ramirez begins to create a final exam for a linear algebra course using the following two problems:

1. Find all solutions to $x + y + z = 2$, $x - y - z = 3$, and $2x - z = 0$ by encoding this system in matrix form and row-reducing the associated augmented matrix.
2. Given four vectors x, y, z, and w suppose that $\{x, y, z\}$ forms a linearly dependent set, $\{y, z, w\}$ forms a linearly dependent set, and that $\{x, w\}$ forms a linearly independent set. Is it possible to determine the dimension of the span of $\{x, y, z, w\}$? If so, what is the dimension? If not, why not?

Professor Ramirez identifies the knowledge dimension of the first task as **procedural** and the cognitive process dimension as **apply** because the task requires students to execute a known algorithm in a specifically given situation. The second problem is assigned a type of **conceptual/analyze**, due to the need for students to conceptually understand the ideas of independence, span, and dimension, and to analyze the possible cases that can arise in this situation. Professor Ramirez decides to have the next two problems on the exam be of type **factual/remember** (asking students to state the definition of an eigenvalue) and **procedural/create** (asking students to create an example of a non-row-reduced 5×5 matrix that has rank equal to 3, and explain why their answer is correct), in order to achieve a broad spectrum of assessment.

Vignette 2

In a graduate-level complex analysis course, Professor Granger has a learning outcome for students to develop an understanding of the fundamental theorem of algebra. The final exam consists of a set of problems to be completed during class and a take-home component. Professor Granger observes that none of the in-class problems are of cognitive dimension **evaluate**, and none of the in-class problems are of knowledge dimension **metacognitive**. Thus Professor Granger assigns the following question as the take-home component of the final exam in order to include these two missing dimensional components:

> In this course we have seen three proofs of the fundamental theorem of algebra (FTA), which use (a) the maximum modulus principle, (b) Liouville's theorem, and (c) an argument using winding numbers. Sketch the main ideas for each of these three proofs, and for each proof list the topics from this course that are required to complete the proof. Choose the proof that you believe should be the first proof of the FTA that graduate students are exposed to, and write a two-page essay providing an argument supporting your choice. Your argument should explicitly include a consideration of the mathematics involved in the proofs.

Discussion: Professor Ramirez uses the taxonomy table when writing the exam to ensure that students are evaluated on a broad range of knowledge types and cognitive processes. Rather than plan out the entire exam based on "filling in the checkboxes," Professor Ramirez begins by writing a pair of problems that are of different flavors and evaluates them using the table. Following this, clusters of gaps in the table provide

1 It is critical to clarify that the evaluation described here is not intended to be a formal psychometric evaluation of items such as those used in educational research. Rather, this taxonomy provides one framework by which teachers can conduct informal evaluations of summative assessments.

inspiration for further problems. On the other hand, Professor Granger writes the complete in-class exam for the graduate students before evaluating the problems. After identifying that one large gap exists on the exam for each dimension, Professor Granger creates a take-home component to address these gaps rather than rewriting the in-class problems. If there had not been a take-home component for the exam, Professor Granger could have decided whether or not to include metacognitive and evaluative items on this assessment. The decision that Professor Granger ultimately made was motivated both by the gaps in the assessment items and the presence of a student learning outcome that would be well-served by a metacognitive and evaluative assessment item.

Practical tips

- Summative assessments should be based on student learning outcomes for a course. Use these learning outcomes as a guide to create summative evaluations of students.
- When using the revised taxonomy to evaluate assessments, use the keywords in the above lists to identify the knowledge and cognitive process level of each problem.
- The two-dimensional framework should not be viewed as prescriptive, but rather as a means by which instructors can evaluate the breadth of their summative assessments.
- The two-dimensional framework is one of many methods for evaluating summative assessment items. If other taxonomies or evaluation methods are more effective in a given situation, then they should be freely used instead.

In the next section we present various assessments that further support Steen's principles.

AP.4. Assessments that promote student communication

Communication and teamwork skills are important for mathematics students. Many aspects of verbal communication and collaboration arise as components of classroom practices, and these also easily fit into both formative and summative assessment. The use of writing assignments can be used for their own sake, to develop student written communication skills, and to assess student learning outcomes that reflect student behaviors, practices, and beliefs.

In this section we describe three types of assessments that support student communication, namely writing assignments, oral presentations, and group projects. These are all assessment tools that are examples of a large collection of activities that constitute "homework." The relationship between homework and assessment is complex because a primary concern of instructors at almost every level is getting students to do the homework. No assessment approach can overcome a lack of effort, but appropriate use of feedback and assessment can raise participation rates in homework and as a result increase student learning.

AP.4.1. Writing assignments

There are many types of writing assignments that can be used in mathematics courses for a wide range of purposes (Bean, 2011; Braun, 2014; Crannell, LaRose, and Ratliff, 2004; Meier and Rishel, 1998; Montgomery and Stuffelbeam, 2014). Assignments such as reflective essays, expository essays, critical analysis of texts, biographical essays, and large-scale course projects have all been used successfully by many mathematicians. Via vignettes, we provide two examples of the use of writing assignments: a reflective essay designed for formative assessment of student mathematical practices and a critical analysis of a reading assignment designed to evaluate student writing skills.

Vignette 1

Professor Germain wanted students to spend more time reflecting on their performance in the course, and specifically wanted students to reflect on their progress toward the student learning outcomes for the class.

When Professor Germain asked the students during class how they felt about their progress, the professor learned that only one student was aware of the learning outcomes as stated in the syllabus. For their next homework assignment, Professor Germain assigned the following reflective essay:

> Read the learning outcomes in the syllabus. Discuss the progress you have made toward these learning outcomes. How have you been successful in reaching these outcomes? How have you not been successful in reaching these outcomes? Has your development toward these learning outcomes had any impact on your work in other contexts, e.g., other classes, jobs, etc? Discuss each of these student learning outcomes with mathematical content examples. This assignment should be: typed, three full pages, 12-point Times New Roman, double-spaced, 1-inch margins.

After the students turned in their essays, one of them asked how the essays would be graded. Professor Germain had not thought about this, and decided that any student who had written a complete and thoughtful response to the prompt that was free from grammatical and typographical errors would receive full credit. While the students accepted this, Professor Germain realized that in the future he needed to create and share a grading rubric.

Vignette 2

Professor Robinson is teaching a real analysis class for mathematics majors. The course is structured using a combination of lecture and small group work. For the class days with small group work, students are required to read a textbook section prior to class. Though the book was chosen based on its accessibility, many students have complained that the book is difficult to read. Despite in-class discussions about reading strategies, it is clear that the students struggle making sense of what they read. In order to dig deeper into the situation, Professor Robinson assigned the following two writing assignments:

> **Prompt #1:** Write a critical review of [specific sections of the textbook]. Imagine that you are writing your review for a journal for undergraduates in mathematics and the sciences. Respond positively to some things and negatively to others; justify your opinions and provide detailed explanations for your claims. Keep in mind that this is a review for a mathematical publication, so you will be graded on both the quality of your writing and on the mathematical depth and mathematical style of the chapter. This essay should be typed, 5–6 pages, double-spaced, and 12-point Times New Roman font. We will complete an in-class peer review of your essay on the day it is due.
>
> **Prompt #2:** Revise your critical review of the textbook sections based on the feedback you received during peer editing. This essay should be typed, 5–6 pages, double-spaced, and 12-point Times New Roman font. You must turn in both your original version (with comments) and your revised version of this essay.

Discussion: Reflective essays are often most effective when a specific, directive prompt is provided to students. For example, Professor Germain's prompt ensured that (a) students read the learning outcomes, and (b) students had plenty of questions they could start to answer in response. In receiving a set of specific questions, the students' attention is more focused in the direction that Professor Germain intends. Otherwise, the student responses might have gone in unintended directions.

On the other hand, when asking students to develop a critical analysis of a textbook, website, video, etc., it is often most effective to provide students a context for the review rather than direction. For example, Professor Robinson's prompt informed students that the audience for their review is other undergraduates in the sciences and that students must have mathematical content in their essay. However, Robinson did not direct students' attention to specific features of the textbooks. This provides students freedom to identify, in their first essay, features that were notable to them and then take into account the opinions and feedback of other students for their revised version. With critical analysis assignments of this type, it is not uncommon for students to change their opinion as they revise their initial essay and develop their final draft.

Practical tips

- Grading rubrics for written work can be found at most campus teaching and learning centers, but these are not always well-suited for mathematical writing. Sample grading rubrics/checklists for mathematical writing, and discussion of related issues, have been developed by various mathematicians (Braun, 2014; Crannell, LaRose, and Ratliff, 2004; Meier and Rishel, 1998).
- For long papers with substantial mathematical content, it can be useful for students to learn LaTeX, a typesetting system. However, for many writing assignments in mathematics courses, standard document preparation software will suffice.
- As shown in the writing prompts above, many questions from students can be avoided by providing clear instructions.
- Many students do not like peer reviews because they feel they do not receive rich feedback. By mentioning this common problem at the beginning of the peer review time, and by requiring students to use the grading rubric when doing their reviews, the quality of the reviews can often be improved.
- When revising work, students should be encouraged to go beyond mere copyediting. Students often find the experience of going through a substantial revision process to be both difficult and rewarding.
- Revision of particular assignments is a good example of Steen's first principle that assessment is a cycle. In addition to the local example of a cycle within the particular assignment, students should be encouraged to see how practice in writing creates long-term alignment with course learning goals.

AP.4.2. Oral presentations

Student presentations can take many forms. In the simplest situation, students can be asked to come to the board or document camera to provide a short explanation of a solution to a problem. Often in courses that use inquiry-based learning (as discussed in the Classroom Practices chapter), students work in small groups and give presentations about their progress, challenges they encountered and resolved, or issues that remain unresolved. In these situations where students present somewhat informally, the presentations serve as formative assessment. For example, in these cases students may be expected to present for participation grades, but not to have an evaluation of the quality of their presentation. At the other end of the spectrum, students may prepare a presentation that is 10-15 minutes in length on a topic in the course, subject to summative assessment using a rubric. These ideas are highlighted in the following vignette.

Vignette

Professor Bassi is teaching a course in Euclidean and hyperbolic geometry. The students in this course are mathematics majors or minors and half of them are 7-12 prospective mathematics teachers. Professor Bassi decides that after the class has collectively covered some of the fundamental content in the course, students will take turns giving 10-minute presentations about topics of their choice. Students will be graded based on five criteria:

1. Clarity of verbal communication
2. Presentation structure
3. Effective use of chalkboard/whiteboard, slides, or instructional technology
4. Mathematical depth
5. Mathematical style

Professor Bassi provides students with a list of suggested topics along with a rubric.

Discussion: Students asked Professor Bassi for clarification regarding "depth" and "style" in their mathematics. This led to a class discussion regarding the difference between deep mathematics presented poorly and simple mathematics presented clearly. Students had not considered the idea that a clear presentation could be viewed as inadequate. Several class discussions were needed to clarify the balance between the depth of the mathematics at hand and the quality of the style of presentation, e.g., the use of sufficiently complicated motivating examples, reasonable "sketches" of proofs rather than line-by-line details, etc.

Practical tips

- The rubric should include components for the presentation and for the quality of the mathematics under discussion. Because students are often not familiar with how presentations in a mathematics course will be graded, it is important to provide a grading rubric in advance.
- Students often ask for feedback on their materials prior to the presentation. Thus, it is useful to provide a deadline for when such feedback requests must be submitted.
- It is important that students begin preparing their presentations early, so that they can ask for help if they have difficulty with unfamiliar content.

AP.4.3. Group projects

Group projects are excellent assessments when instructors want to monitor student performance on tasks not suited for a timed exam, to promote active student interaction with classmates, and to provide students with experience working with non-routine problems. Good problems for group work often have the characteristic that they encompass content from recent class meetings as well as knowledge of other concepts addressed earlier or in previous courses. Such problems typically are not appropriate for a timed exam or quiz, because they frequently involve significant conceptual problem-solving skills that draw more broadly on the students' mathematical backgrounds. This type of assignment can have a positive impact on students, because it requires a more authentic engagement with mathematics than straightforward computational exercises. It also requires students to persist through the process of making sense of the problem and attempting multiple solution strategies, as suggested by Black and William (2009). We illustrate these notions in the following vignette.

Classroom vignette: Group projects

Professor Herschel wants students in calculus to gain experience working together to solve problems requiring more than the application of a specific technique illustrated by an example in the textbook. Herschel assigns four group projects using problems such as, "Show how to find the minimum value of the area of the region under the curve $y = x + 1$ from $x = a$ to $x = a + 1.5$ for all $a > 0$."

Professor Herschel assigns students to groups of four and provides the following directions and grading structure for the group projects during the course.

> This problem is to be completed in your group; each group member will receive the same grade. There will be four problems during the semester. Each group member is responsible for writing the final report for at least one of the four assignments. The report will have three parts, defined and graded as described below. If a part is missing it will receive zero points; based on this rubric a correct solution alone is worth at most four out of ten points!
>
> **Part 1.** (3 points) Describe the problem, including challenges you encountered. Do not just restate the problem. Instead, show that you understand what you are trying to do, but do not show how you solved or attempted to solve the problem. That comes later. This will require a paragraph—that is, more than a sentence and less than a page.

Part 2. (3 points) Describe how your group worked on the problem. Describe what you tried, including both things that worked and false starts. This is a description of your strategies, not a point-by-point listing of every thought that occurred to you. This will also take more than a sentence or two, but less than a page.

Part 3. (4 points) Provide your solution to the problem. This should be accurate, carefully written, and concise.

In each of your responses, follow these guidelines:

- Use complete sentences, even if this is something you have not done in the past when writing a solution to a mathematics problem.
- Work on these problems only with members of your group. Everyone is expected to contribute by working hard and discussing their thoughts. You must include this statement at the end of your report along with signatures:

  We have neither requested nor received any help from individuals outside our group on this problem. Each of us has contributed to the work on this problem, and we will not allow anyone to sign this paper who has not contributed in some way.
- Discuss any problems about your group work with your instructor.
- Provide complete explanations of how you found your answers. An answer without support will receive little or no credit.

Discussion: Instructors should be cautious about leading students too much on such tasks. One can gain considerable insight into students' mathematical reasoning by judiciously providing hints and indications about whether or not students are on the right track before assignments are due. In class, different groups can present their partial or complete solutions so that the entire class has a chance to practice mathematical discourse in a situation where there is no external confirmation of the correctness of any solutions. The instructor will gain considerable insight into student reasoning by listening and questioning during this student-centered classroom discussion. Students will learn how to problem-solve in the absence of a solution template or final answer, a valuable insight into the nature of mathematics.

Practical tips

- When designing group projects it is important to emphasize to students the reason for requiring them to work collaboratively. Many students are resistant to having their grade depend on others. An instructor may increase student buy-in and enthusiasm by explaining that group work reflects the reality of many job expectations in business, industry, and government, and also by explicitly connecting group work with student learning outcomes listed in the course syllabus.
- If a particular group appears dysfunctional, be willing to revise the group assignments mid-course. However, make every effort to avoid doing this, so as to maintain coherence in how the groups are handled for all students.
- Some instructors allow students to choose their own groups, while other instructors assign groups. Either approach can be effective, so instructors should make this decision based on their own preference and their understanding of their students.
- If groups are allowed to switch members throughout the course, the responsibility of serving as the "lead writer" for each group should be evenly distributed among the students throughout the course. A lack of policy in place to handle distribution of workload can be a cause for discord and tension among members of a group.

- The number of group projects can be adjusted for various courses. As with most changes to teaching, it is a good idea to start small. You can always incorporate more group projects as you gain experience and confidence.

AP.5. Conceptual understanding: What do my students really know?

It is common for students at all levels of mathematics to be able to correctly complete an arithmetic or symbolic procedure, and yet be unable to explain the mathematical concepts underlying the procedure or the viability of the computation. This distinction can be examined by identifying the components of mathematical understanding referred to as **conceptual understanding** and **procedural fluency**. The typical mathematics assessment consists of a large number of procedural problems, in large part because it is more challenging to construct appropriate and meaningful problems that evaluate conceptual understanding. It is also more time-consuming to grade conceptual questions. In this section we discuss conceptual understanding and concept inventories, and then turn our attention to how the idea of a concept inventory can serve as a springboard for creating summative assessments of student conceptual understanding.

AP.5.1. What is conceptual understanding?

In practice, the term “conceptual understanding” is often used to refer to skills involving the ability to explain why something happens mathematically, using logical reasoning as opposed to empirical evidence. The definition of conceptual understanding and its relationship with other dimensions of mathematical knowledge, particularly procedural fluency, has been debated and discussed in the mathematical sciences community (see e.g., Baroody et al., 2007; Star, 2005). Certainly conceptual understanding and procedural fluency as well as other mathematical skills are strongly interrelated, and there may be many specific skills or types of knowledge that integrate both procedural fluency and conceptual understanding (see e.g., Hiebert and Lefevre, 1986; NRC, 2001).

In fact, conceptual understanding has been identified as one of the critical components of learning mathematics (NCTM, 2000; NRC, 2001), and as such, considerable efforts have been made over the past two decades to develop this notion. Mathematics educators, mathematicians, and curriculum developers have made substantial efforts to develop high-quality assessments of conceptual understanding. Different mathematical tasks engage students in different types of reasoning and thereby result in different types of learning. For example, some tasks engage students in memorizing or practicing specific procedures, while other tasks engage students in complex reasoning and sense making. The latter can be assessed via concept inventories.

AP.5.2. What are concept inventories?

As conceptual understanding has become more commonly recognized as a critical component of mathematical understanding, mathematics educators have increasingly developed **concept inventories** as a way to assess students’ conceptual understanding in different content domains. Concept inventories are examinations that test basic but fundamental concepts within a particular subject. They do not test computational skills, nor do they aim to address everything that a student might learn in a course, but rather they aim to test concepts that are necessary (but not necessarily sufficient) for mastery of course material. Traditionally, concept inventories were developed for research purposes in order to compare the effect on gains in conceptual understanding in a particular content domain that occur under different teaching strategies. For example, an instructor might administer an exam at the beginning and at the end of the semester, and compare the effect of different teaching strategies on conceptual understanding.

While concept inventories were originally developed with research purposes in mind, this type of assessment can also be used by classroom teachers and by departments to assess the extent to which students have gained conceptual understanding in a particular course. Concept inventories seek to identify which concepts students understand and which misconceptions are a barrier for student understanding. However, not all concept inventories have been validated, and just as with other standardized exams, departments should be cautious about making high-stakes decisions without first rigorously testing the validity and reliability of these instruments with their student population (Bagley, 2016; Gleason et al., 2015).

The first well-studied concept inventory was the Force Concept Inventory (FCI) in physics (Hake, 1998; Halloun, Hake, Mosca, and Hestenes, 1995; Hestenes, Wells, and Swackhamer, 1992). Other concept inventories exist in physics (e.g., Halloun and Hestenes, 1985a; Halloun and Hestenes, 1985b), chemistry (e.g., Mulford and Robinson, 2002), and biology (e.g., Klymkowsky, Underwood, and Garvin-Doxas, 2010; Smith, Wood, and Knight, 2008). In mathematics the Calculus Concept Inventory (CCI) (Epstein, 2013) is the first concept inventory widely tested on a large student population. Other concept inventories exist for precalculus (Carlson, Oehrtman, and Engelke, 2010), elementary algebra (Wladis et al., 2017a, Wladis et al., 2017b), differential equations (Hall, Keene, and Fortune, 2016) and group theory (Melhuish, 2015).

Here are two examples of assessment items that measure conceptual understanding.

Example 1: Figure 2 shows a problem from the Precalculus Concept Assessment (PCA).

Assume that water is poured into a spherical bottle at a constant rate. Which of the following graphs best represents the height of water in the bottle as a function of the amount of water in the bottle?

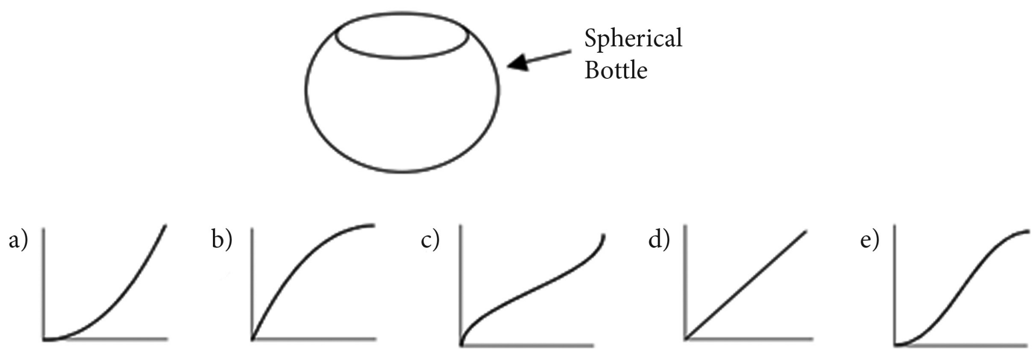


Figure 2. From the Precalculus Concept Assessment (Carlson et al., 2010).

This question tests the extent to which students can correctly employ covariational reasoning—that is, the extent to which students can explain how one variable will change as the result of a change in another variable. This type of reasoning is central to algebra and calculus, but research suggests that students struggle with this concept (Carlson, Jacobs, Coe, Larsen, and Hsu, 2002). While instructors who teach college-level mathematics may not use the term covariational reasoning to describe this type of student reasoning, they will almost certainly be familiar with many of the misconceptions that students employ when solving such a problem.

This type of concept inventory item provides instructors a tool to determine which students can employ covariational reasoning and to identify potential misconceptions for those students who struggle with this type of reasoning. For example, both choices a) and d) are typically chosen by students who fail to recognize that the rate at which the height changes in relation to the volume varies over time. Another reason students often choose option a) is because they perceive that the graph's upward shape indicates the height of the water increases more and more. While these students recognize the volume increases as height increases, they are unable to distinguish between an increasing graph and an increasing slope on that graph. Both types of students may need additional tasks that force them to confront these two different features of a graph and to make sense of the different information the features convey.

Students who chose option b) often reason that this graph represents the shape of the left side of the bottle because they do not understand that the graph should depict the rate at which the height increases as the volume increases. These students perceive the graph as a picture of the static object modeled. Students who chose option e) often perceive that the water rises slowly at first, then more quickly, and then slowly again. It may be that these students reach this conclusion because they confuse the rate at which the height of the water increases with the width of the bottle at a given volume. These two types of students would benefit from tasks that challenge them to explore and generalize how two co-varying quantities change over time and to relate these patterns to graphs that depict the co-varying relationship. For example, students could be presented with a task involving various composite objects such as a cylinder, a downward pointing cone, or an upward pointing cone, and asked to calculate the height at various specific volumes. Furthermore, the students could compare how the height changes as the volume increases, then construct a corresponding graph of volume versus height for each shape. An interactive applet or a computer algebra system may be the most efficient way to do this. Students could then be asked to compare the rate of change of the height with respect to the volume as depicted in the various graphs.

Example 2: Consider this question that is similar in format to those on the Calculus Concept Inventory (CCI) (Epstein, 2013):

> The derivative of a function is negative everywhere on the interval $x = 2$ to $x = 3$. Where on this interval does the function have its maximum value?
>
> a) At $x = 2$.
> b) We cannot tell if it has a maximum because we do not know where the second derivative is negative.
> c) Somewhere between $x = 2$ and $x = 3$.
> d) At $x = 3$.
> e) It does not have a maximum because the derivative is never zero.

This question assesses the extent to which students can connect characteristics of the derivative to the actual behavior of the underlying function.

Students who choose options b) and e) are likely relying on procedural rules rather than reasoning about characteristics of the derivative and the shape of the underlying function. These students would benefit from classroom tasks that require them to connect features of a function to its derivative including working backwards from information about a derivative to determine features of the underlying function. For example, students could be asked to draw the derivative of a function based on the function's graph, or they could be asked to draw a few possible graphs of an underlying function based on a graph of its derivative.

AP.5.3. Using items from concept inventories

One way to use concept inventories is to select a few tasks to use during an in-class activity. As the students work collaboratively on the tasks, the instructor might walk around the room and ask students to explain their reasoning. The instructor might also facilitate a class discussion during which students explain their reasoning. Such practices could help identify and correct student misconceptions that might impact successful course completion, success in subsequent mathematics courses, and students' ability to apply the concepts to "real life" situations.

The following vignette exemplifies how this process might be used in an elementary algebra class.

Vignette

Professor Jackson notices that students make certain common errors on procedural problems. For example, when students are asked to substitute -3 in for in the expression $2 - x^2$, many students write $2 - 3^2$ instead of $2-(-3)^2$. In order to better understand why students do this, Professor Jackson gives students the following problem:

Consider the expression $\frac{1}{a}-a^2$.

Perform the following substitutions (no need to simplify afterwards!):

a) Substitute -2 for a.

b) Substitute x^2 for a.

c) Substitute $\frac{1}{y}$ for a.

Professor Jackson then engages students in both small group and whole class discussions about their strategies for solving the problem.

Discussion: Students frequently make the following mistakes on the three substitutions in this problem:

a) $\dfrac{1}{-2}-2^2$ b) $\dfrac{1}{x^2}-x^2$ c) $\dfrac{1}{y}-\left(\dfrac{1}{y}\right)^2$

Students make the first mistake above because they forget to write the negative sign in front of the 2 or they mistake the subtraction sign for the negative sign and think they do not need to include the negative sign for . This is an opportunity for the instructor to articulate an important algebraic concept related to substitution: substitution is the process of replacing a variable with a numerical value or an expression and all the algebraic structure outside the variable remains unchanged by the substitution.

The following items could be used as follow-up assessments to determine whether or not a student understands the concept of substitution.

Item 1: Which of the following expressions is the result of substituting -2 into the expression $\frac{1}{a}-a^2$?

a) $\dfrac{1}{2}-2^2$ b) $\dfrac{1}{-2}-(-2)^2$ c) $\dfrac{1}{-2}-2^2$ d) $\dfrac{1}{2}+2^2$

Item 2: What happens to the negative sign in the expression $\frac{1}{a}-a^2$ if we substitute -2 for a?

a) Nothing—when we substitute something for a in this expression, the rest of the expression remains the same.
b) It is replaced by the negative sign that belongs to the -2.
c) It is canceled out by the negative sign that belongs to the -2.
d) Nothing—but we drop the negative sign from the -2 because there is already a negative sign in the expression.

These types of questions can be used as both formative and summative assessments. As with any classroom assessment, instructors can create their own concept inventory questions tailored to their own students' misconceptions. The fundamental procedure followed by researchers who have developed concept inventories mimics what instructors often do as a typical part of classroom practice. They create an open-ended question they believe pinpoints important underlying concepts, and they use it for class discussion or as a short written assignment to elicit common student answers. Based on these results a multiple-choice question could be developed and used in a subsequent semester or on a subsequent assignment in the same semester.

For example, an instructor could present one of the two items above as an open-ended question during class, solicit student answers and write them on the board, and ask students to vote on which answer they think is correct before proceeding with a discussion about the relevant substitution concept. The most common incorrect student responses could be used as a basis for the multiple-choice question on a follow-up exam.

This example demonstrates how conceptual understanding might be assessed both formally via exam questions as well as informally via in-class activities, illustrates how formal and informal activities inform one another, and highlights the relationship between conceptual understanding and procedural fluency.

AP.6. Assessment in large-enrollment classes

Many higher education institutions have large-enrollment classes. Institutions with graduate programs in the mathematical sciences often connect small recitation sections led by graduate teaching assistants to such classes. Two assessment techniques that are particularly helpful in large-enrollment classes are online homework systems and classroom polling techniques.

AP.6.1. Online homework systems

In the spring of 2009, the American Mathematical Society (AMS) surveyed 1,230 U.S. mathematics and statistics departments about their experiences with online homework systems (AMS, 2009; Kehoe, 2010; Lewis and Tucker, 2009). The survey respondents indicated the most important benefits of online homework systems were (a) the immediate feedback provided to students, (b) the opportunity for students to attempt an exercise multiple times, and (c) the reduction in grading duties. On the other hand, survey respondents indicated disadvantages to online homework were (a) the inability for students to show their work, (b) the limited types of questions that can be effectively evaluated online, and (c) students' frustrations with the systems. Reducing the amount of time an instructor spends grading homework allows the instructor to increase the time spent on more meaningful instructional activities and assessment that can better illuminate conceptual understanding and misconceptions. Students' frustrations might be alleviated by engaging them in frequent conversations about the value of online homework and helping them discover support mechanisms such as links to various online tutorials that provide further explanations.

Various researchers (Bonham et al., 2001; Doorn et al., 2010; Hauk and Segalla, 2005; Malevich, 2011) synthesize the pros and cons of online homework systems. Additional advantages include the randomization of variables and parameters that can mitigate cheating and save departments money on hiring graders. Other disadvantages include (a) the inability to provide students with reasons why responses are incorrect (b) the inability to prevent cheating, and (c) the additional costs incurred by students if they must purchase access to these systems.

As of 2017, the most commonly used online homework systems, in alphabetical order, are ALEKS, MAA WeBWorK, MyMathLab/MyStatLab, MyOpenMath, and WebAssign. These systems are typically showcased at the national conferences of the MAA, AMATYC, and AMS, which are good places to explore each system and compare their features. Below is a brief description of each of these online homework systems.

ALEKS (McGraw-Hill Education): ALEKS (Assessment and Learning in Knowledge Spaces) is based on knowledge space theory (Doignon and Falmagne, 1999, 2011; Falmagne et al., 1990), a field fueled by funding from the National Science Foundation in the 1990s. It uses artificial intelligence to assess a student's knowledge of concepts and procedures. When a student takes an ALEKS assessment, they typically complete 20 to 30 problems to determine their current knowledge of course content. Each question ALEKS chooses is based on the student's answers to previous questions. The system displays a pie chart illustrating which topics the student has mastered and which topics they are ready to learn. The student chooses from a list of topics they have not yet mastered but are ready to learn. ALEKS provides practice problems on that topic and the student may request a detailed solution to each problem. Once the student answers a sufficient number of practice problems correctly without access to the solution, ALEKS determines the student has mastered that topic, and the student then chooses another topic. ALEKS is available for use in a wide variety of K–12 and undergraduate mathematics courses.

MAA WeBWorK (MAA): WeBWorK is an open-source online homework system developed via funding from the NSF and includes an Open Problem Library (OPL). Problems are available for a wide variety of mathematics courses ranging from college algebra and calculus to differential equations, linear algebra, and complex analysis. Instructors can select problems from the OPL or write new problems themselves. Like other online homework systems, WeBWorK provides students with immediate feedback and can be linked to learning management systems.

MyMathLab/MyStatLab (Pearson): MyMathLab and its equivalent in statistics, MyStatLab, offer videos, quizzes, and homework assignments. MyMathLab can be linked to various learning management systems such as Blackboard, Canvas, and Moodle, and assignments can be made from the selected course textbook or chosen from other MyMathLab courses. MyMathLab offers adaptive learning skill building exercises in select courses. Each question in a skill building exercise is based on the student's previous answers. MyMathLab also offers ”workspace assignments” that allow a student to work through an exercise step-by-step and receive immediate feedback at each step. Students physically write out their answers and the system uses handwriting recognition software to evaluate them. MyMathLab and MyStatLab are available for use in a wide variety of K–12 and undergraduate mathematics courses

MyOpenMath: MyOpenMath (www.myopenmath.com/) is designed for self-study and online courses in developmental mathematics. It takes advantage of open source materials to provide access to students who cannot afford traditional texts or software licenses.

WebAssign (Cengage): WebAssign was developed in 1997 at North Carolina State University and purchased by Cengage in 2016. However, the selection of textbook titles that utilize WebAssign come from a wide range of publishers. It offers videos, interactive content, and tutorials to aid students on assignments. Instructors can build assignments by selecting exercises from the textbook or by creating their own questions. An analytics tool allows instructors to identify problems and topics with which students had the most difficulty. Instructors can collaborate with colleagues by sharing homework questions and assignments. As with MyMathLab, WebAssign can be linked to various learning management systems like Blackboard, Canvas, and Moodle. The types of mathematics courses for which WebAssign can be used range from basic algebra to calculus, discrete mathematics, and ordinary differential equations.

The most recent version of the *Guidelines for Assessment and Instruction in Statistics Education* (GAISE, 2016) from the American Statistical Association notes the increasing trend of gaming as an entertainment source for college students. Gamification in educational settings is the use of game design elements in non-game situations with the goal of increasing student engagement (Attali and Arieli-Attali, 2015; Sandusky, 2015). An exciting new enhancement to WeBWorK for use in calculus courses is a gaming mechanism developed by Goehle (2013) that utilizes the common video game system feature of “levels.” In Goehle's system students earn “experience points” for every correct homework problem, and they are able to move up to the next level when they have accumulated a sufficient number of points. The lowest level is “Calculus initiate” and the highest is “Calculus professor.” As the levels increase, so do the number of points required to advance to the next level.

The following vignette illustrates the benefits of online homework systems.

Vignette

Professor Phillips teaches at a private, comprehensive university. Admission is competitive, and the school is well regarded, but the endowment is not large and most revenues come from undergraduate tuition dollars.

Competition for good students requires the university to offer more and more merit-based scholarships, so the budget is tight and the administration is looking for economies of scale in all aspects of the operation. For the Department of Mathematics and Statistics, this means their service courses such as business calculus and introductory statistics, traditionally taught in classes of about 25 students, will now be taught in classes of 75–90. There are no graduate students in quantitative fields to serve as teaching assistants, and there is limited support for undergraduate graders.

Professor Phillips is discouraged, as teaching small classes was what attracted him to the university in the first place. The issue at hand is how to make the best of the move to larger class sizes. Professor Phillips has traditionally collected homework in hard copy, graded it carefully, and quickly returned it to students in order to provide them timely, detailed feedback for improvement. Recognizing this will no longer be possible, Professor Phillips decides to investigate how to take advantage of online homework systems so he can continue to best serve his students.

The professor is pleasantly surprised to find the developers of these systems seem to have thought seriously about using evidence-based practices in assessment. Because the reduction of direct connection from instructor to student is a clear cost of the change to online systems, Professor Phillips is eager to find a system that makes it possible for him to carefully monitor student progress and intervene as needed with the whole class as well as with individuals.

After significant discussions with colleagues in the department, they agree that one instructor will hold office hours devoted to technical issues with the chosen system. They also agree that a student who scores at least 85% on an online assessment will be credited with 100% in order to proactively address potential student complaints about round-off errors and technical issues. During conversations with colleagues from other institutions, Phillips discovers that gaming aspects available with some online systems also help to mitigate student frustrations. He decides to integrate online games into an introductory statistics course where students collect and analyze data from virtual reality environments to determine factors that impact winning in a game. Incorporating the game promotes a higher level of student engagement and results in students mastering concepts outside of class. In-class discussions then serve as formative assessments he uses to tailor subsequent classes to enhance student learning.

AP.6.2. Classroom polling systems

As discussed in section CP.1.9 of the Classroom Practices chapter, classroom polling systems can be incorporated into mathematics courses in a variety of ways, ranging from using electronic "clicker" systems to voting by show of hands. Here is one example of how this might be used, and Cline and Zullo (2011) offer other suggestions.

Classroom vignette: Introducing a classroom polling system

Professor Ordinal teaches at a public state university that draws most of its students from the nearby urban area but has a growing number of students from other parts of the U.S. and other countries. The school has a college of engineering, and Professor Ordinal teaches one large section of an engineering mathematics course each semester. This is essentially a fourth semester course for STEM students, following a three semester calculus sequence, and it is meant to introduce key concepts in linear algebra and differential equations needed in upper-level coursework. Each semester about 150–200 students enroll in Professor Ordinal's section, in which instruction consists of large lecture meetings and weekly recitation sections led by teaching assistants.

Students' perceptions of the course are strongly negative, and they consistently give poor evaluation ratings. They have a sense that the course is designed to serve as a last effort to weed out students who are perceived to not be strong enough to major in STEM fields. It has reached a point where the engineering college is threatening to develop their own alternative course and no longer require the mathematics department's course. Professor Ordinal is aware of this perception and would like to increase student engagement while maintaining high standards. A full flipped classroom approach seems impractical given the enrollment.

Professor Ordinal decides to utilize a classroom polling system. After some initial trial and error, Ordinal develops some carefully crafted questions tied to the key learning goals and intended to assess mastery of concepts and encourage class discussion. Professor Ordinal ensures students have sufficient time to formulate answers but not enough time to become distracted.

Discussion: While a primary purpose of such systems may be to increase student engagement, the system can be an effective tool for formative assessment in real time and can play a role in summative assessment as well. This is particularly true in large sections like Professor Ordinal's.

Classroom polling is hardly novel and was used well before technology made compiling responses relatively quick and easy. "How many think the conjecture is true? If so, raise your hand!" This is the sort of throw-away line that instructors have inserted into lectures since the beginning of time. Experienced instructors know that this old-school approach, while perhaps better than nothing for encouraging engagement, makes it difficult to record student responses in a timely manner, can result in high non-participation rates, and amplifies the peer-pressure factor that may cause students to respond in the same way as their peers. Perhaps most important, this technique encourages rapid responses rather than thoughtful reactions.

Practical Tips

- Consult your campus academic technology group before choosing a polling system. They typically have the expertise to answer specific questions like whether polling participation can be incorporated into a classroom management system. Many institutions have an expert who will ensure you do not reinvent the wheel and are able to successfully launch your system.
- Consider online options such as pollanywhere, if the student demographic leads to a reasonable assumption that most or all students in the class have a smartphone.
- Determine what other systems might already be in place at your institution. There is also considerable advantage in using systems with which students are already familiar.
- If polling is to be incorporated into the course grade, learning goals for the course must include participation and the rubric for grading must be clear.

AP.7. Assessment in non-traditional classrooms

AP.7.1. Assessment in online courses

Given that the higher education enterprise must respond to the demand for online learning, instructors find themselves teaching online courses and rethinking assessment of both teaching and learning (Hewson, 2012). In order to ensure online students achieve at a level comparable to students in face-to-face settings it is vital for educators to reexamine basic notions of both formative and summative assessment in online courses (Stewart, Waight, Norwood, and Ezell, 2004).

Online learning is a space where the principles of constructivist, learner-centered, and authenticity-based education can be created (Lesnick, Cesaitis, Jagtiani, and Miller, 2004). The assessments that worked perfectly in a face-to-face setting may need to be re-conceptualized, tweaked, or even replaced in an online environment. The issue of validity and dishonesty related to assessment in online courses should be examined carefully in the design of courses. Online education technology allows a number of assessment tools, such as discussion boards, surveys, and online discussion groups, all of which can be modified into formative or summative assessments to document student learning based on the course objectives. Creation of authentic and effective assessment, both formative and summative, is possible with the use of online education tools.

In the online environment, the lack of physical contact between instructors and students leads to different assessment techniques. In the online setting, instructors cannot tell whether a student is in attendance unless he or she is actively contributing something to the class. It is for this very reason that in typical online classes, 10–25% of the course grade is for discussion participation (Anderson and Elloumi, 2004). To prevent cheating and to create a learner-centered environment, assessment of students is typically based on a variety of assignments, quizzes, papers, tests, group projects, and discussions (Jarmon, 1999, pp. 55–63). Students are kept abreast of their grades throughout the class, rather than at some specific junctures during a term. This increased emphasis on continual and alternative assessment methods has great potential to increase the transparency of the learning process and improve learning.

Arend (2006) conducted a study on how course assessment practices relate to learning strategies for students taking online courses in two-year colleges. Learning strategies are defined as specific techniques students use when studying for a class. In this study, Arend finds that most formative assessment variables and general summative assessment variables do not show significant relationships to the learning strategy. Instead, assessment methods such as discussions or papers are significantly related to learning strategies. It is evident that the more a course uses discussions, writing assignments, and papers, the more students use critical reasoning strategies. Conversely, the more a course relies on finals and midterms for assessment, the less time students spend on critical reasoning. The courses in the study used online methods of exams, discussions, written assignments, problem assignments, and experiential activities. Additionally, many larger assignments were broken down into smaller pieces focusing on different aspects and were graded over time. The study concluded that assessment techniques that evaluate multiple dimensions of learning are most effective since they provide opportunities for students to demonstrate and extend their learning. Such constant assessment techniques allow educators to provide feedback and provide students with an opportunity to learn from their mistakes (Kerka and Wonacott, 2000).

Arend (2006) notes that summative assessment and the number of assignments used in a course can be one area of concern. Although multiple, shorter assignments are deemed better than using only a small number of high-stake assignments, there is some indication that the total number of assignments can too high. Some courses in the study used between 50 to 90 assignments in a 15-week period. In a course with too many short assignments, students are at risk of focusing their attention on completion of the assignments, rather than understanding of course material. The literature in general supports the use of exams to document performance of students in an online environment (Hewson, 2012).

AP.7.2. Assessing via technology

Innovative initiatives are beginning to demonstrate the potential of technology-enhanced assessment for integrating formative and summative assessments. The online assessment tools developed to assess students' achievement and progress allow educators to set flexible tests at the required level as well as to measure and record student progress over time. Such systems provide rich feedback for instructors on specific aspects of student performance and support assessment *for* learning and *of* learning. Technology also supports the use of summative assessments for formative purposes, enabling traditional testing methods to be used in more meaningful ways. The use of multiple-choice questions, for example, is most commonly associated with testing the recall of facts with no associated elements of useful feedback or learning interaction. When combined with the use of digital communication tools, it can prompt new ways of activating assessment for learning. Carefully chosen questions answered by learners via mobile devices or electronic systems can be used by educators to identify alternative or comparative understandings and to provide students with real-time feedback.

For effective learning, students need to be actively involved in feedback processes rather than simply serving as passive receivers of information about their progress. Self-assessment has shown to improve learning outcomes through students' reflection on and revision of their own work. Technology-enhanced assessments can support students' active participation in integrated systems of formative and summative assessments.

Technology alone cannot transform assessment practices, and the role of the instructor remains of vital importance in all educational fields. This is particularly important in connecting technology to make assessment more relevant and related to learners' achievements and progress. Digital tools used in online courses that can support integrated assessment practices that are relevant and appropriate to the context, and to the learners, open up new possibilities for more personalized, immediate, and engaging assessment experiences. The use of digital technologies for assessment must support improved assessment practices and preferred educational outcomes. We must acknowledge the complexity of the task and the significant ethical questions raised by the use of digital technologies in assessment.

The online environment offers some unique challenges for assessment, but it also offers opportunities for positive ongoing assessment. For example, one could ask what online techniques can be used to make student reasoning visible. Or, how can rubrics be used effectively to inform the evaluation process in an online class? Are there tools that help create and execute assessment instruments? And, how could academic honesty and ethics be promoted when assessments are taken online?

Self-assessment, goal setting, and motivation: The most effective students generally set personal learning goals, use proven learning strategies, and self-assess their work. Teachers can help cultivate such habits by teaching students to self-assess, to set goals, and to expect students to apply these habits regularly. Teachers who provide regular opportunities for students to self-assess and set goals often report a change in the class culture from students asking, "What did I get?" or "What are you going to give me?" to becoming capable of knowing how they are doing and what they need to do to improve. This is particularly important in online courses, due to inherent challenges to instructors in providing direct feedback on student practices. Naturally coupled with this challenge is that summative assessment strategies must influence students to become motivated to learn. Students are more likely to put forth the required effort when they clearly understand the learning goals and standards, they know how teachers will evaluate their learning, they think the learning goals and assessments are meaningful and worth learning, and they believe they can successfully learn and meet the evaluative expectations.

Quizzes and exams: Quizzes and exams are generally used to measure academic achievement of students. Such tools can consist of multiple choice, matching, and free-response items. Different learning management systems (LMS) make it possible for instructors to adapt the design and deployment of the assessment. Mainstream LMS coupled with a product like *Respondus* enable administration of quizzes 24/7 with safeguards against student collaboration. Also, many LMS are capable of assigning each student random questions, timed tests, password protection, and adaptive release of quizzes.

For non-proctored assessments certain security options can be deployed. The instructor can release the test or quiz one question at a time and not allow students to go back. To reduce the possibility of students getting answers from one another, the instructor can create a pool of questions and make different versions of the same test/quiz. Most LMSs allow instructors to make parallel forms of a test. In addition, it is possible to randomize the presentation of questions on a test as well as the answer options. Using randomization, parallel versions of tests, and synchronous test periods makes it difficult for students to consult with one another on test items.

However, with the presence of computer assisted systems like *Wolfram Alpha*, *Symbolab*, *Cymath*, etc., it is very easy for students to obtain answers to procedural questions quickly, creating potential academic integrity issues. As another option to promote academic integrity with non-proctored quizzes or tests, instructors can use items that require higher-order reasoning, where the answers to the questions go beyond computations that a computer program can do. Overall, it is best to assign a low percentage of the course grade to these non-proctored quizzes or tests. In addition, inclusion of an academic honesty statement at the beginning of tests or quizzes will serve to remind students of the penalty for dishonesty. For example, a statement like the following keeps expectations clear: "I completed this quiz myself, worked independently, and did not consult anyone except the instructor. I have neither given nor received help on this quiz. I understand that academic dishonesty results in consequences as described in the university catalog."

When midterm or final exams account for a significant proportion of a course grade and are completed in the privacy of a student's home, ethical questions about the validity of the results and the legitimacy of the educational program naturally arise. In fact, a growing number of higher education institutions only transfer credit for an online course if a high percentage of the exams are proctored. Awarding meaningful grades requires that instructors take steps to ensure what is known as "academic authenticity," meaning simply that students do not cheat. Authentication of identity is a common concern, and various methods have been developed in response, including palm vein recognition, keystroke recognition, and questions about public record information that only the authentic individual might know in detail (Sandeen, 2013). Because of cost, human proctoring is currently the most popular approach. This is done in various ways. One method is for the learner to be supervised by a proctor such as an instructor or administrator selected by the learner and approved by the home institution. An alternative is for the student to travel to a regional site sponsored by the home institution. A third option is to use companies that specialize in test delivery. These companies provide services such as checking multiple forms of identification, photographing candidates, and videotaping test sessions. Therefore, to give credibility to courses taught at a distance, to facilitate transferability, and to satisfy accrediting agencies on the soundness of programs, real effort must be placed in implementing assessments that preserve the integrity of courses taught online.

In an online class it is easy to add multiple-choice tests to one's Learning Management System, but assessment is much more than that. Assessment involves identifying clear, valid, and appropriate student learning outcomes, collecting evidence that those outcomes are addressed, setting the stage for a dialogue to attain a collective interpretation of the data, and using data to improve both teaching and learning. Assessment can certainly be a tool for accountability, but it can also be an ongoing process for learning. The concept of student-centered teaching involves effective use of both formative and summative assessment regardless of the mechanism for delivering course content.

AP.7.3. Assessment in non-traditional course settings

While assessment is typically discussed in the context of classroom settings, it also plays an important role in non-traditional course settings. Courses focused on service learning, independent study, undergraduate research, industrial projects, and internships all benefit from the thoughtful incorporation of assessment. Because these types of courses vary dramatically from one institution to another, we will focus our discussion on three key themes that should be considered by instructors teaching such courses.

Create student learning outcomes: Even in less formal course situations, students and instructors benefit when there is a clear vision for the purpose and scope of the course. For example, if a student participates in an independent study it is best to establish a target set of topics to be covered with a tentative reading schedule. Similarly, if one of the goals for the independent study is that the students will increase their ability to independently read mathematical texts, this must be clearly communicated with the students so they are attentive to their work in this regard. By creating written student learning outcomes and sharing these with the students, instructors can establish clear goals for students and ensure that students are aware of the desired outcomes for the course.

Formative assessment through written reflections: Non-traditional courses often do not incorporate traditional homework and exam structures. For example, in a service learning course, the content of the course is often engagement in the mathematical community (through tutoring, outreach, etc.) rather than specific topics of mathematics. Similarly, during an internship for which students receive credit, the learning outcomes might focus on experiencing the ways in which mathematics is used in a specific business, industry, or government setting. In these situations, one effective way to conduct formative assessment is through the use of written reflective essays, as discussed previously in this chapter. By providing students with specific prompts, or even by directly requesting that students write about their progress toward meeting the student learning outcomes, instructors can support deep reflection and meta-cognitive growth in students throughout their course experience. In courses that have a larger mathematical content component, such as independent study or undergraduate research courses, it is natural to have students include written updates regarding their mathematical investigations in addition to any verbal updates provided during meetings.

Summative assessment through written portfolios: A natural way to build on the use of written reflections as formative assessment is to have students compile these into an end-of-course portfolio serving as the summative assessment instrument. Instructors might require that students write a 3-5 page essay describing their growth and development in the course and their self-assessment with regard to the student learning outcomes for the course. In independent study and undergraduate research courses, it is appropriate to also include a final written report on the results of the mathematical investigations completed during the course.

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Design Practices

Mathematics instructors today have access to an expansive body of research literature that addresses how students learn, effective teaching methods, and how course context affects students' ability to learn. This research is threaded through the Classroom Practices and Assessment Practices chapters of this document and provides a foundation for this Design Practices chapter. We define design practices to be the plans and choices instructors make before they teach and what they do after they teach to modify and revise for the future. Design practices inform the construction of the learning environment and curriculum and support instructors in implementing pedagogies that maximize student learning.

Key aspects of course and lesson design are identifying goals for student learning, selecting instructional strategies to achieve those goals, and choosing methods to assess student learning. This chapter makes these considerations explicit while acknowledging that great teachers have long engaged in these activities in the construction of their courses.

The chapter is divided into four sections: (1) an introduction to design practices, including questions that can guide design; (2) research-based components of design; (3) opportunities and challenges in designing for student-centered engagement; and (4) a brief introduction to theories of design. Integrated throughout this chapter are seven in-depth classroom vignettes, examples of how design practices might appear from the concrete point of view of a mathematics instructor. While each example is placed to help illustrate a particular point, most examples are much broader and reinforce other points throughout the chapter. As a collection they exemplify design at granularity ranging from a single task to an entire course. Given that the MAA Committee on the Undergraduate Program in Mathematics Guide (2015) provides examples of designing *programs* in the mathematical sciences, we do not provide that level of granularity here.

DP.1. Introduction to design practices

Design practices take many forms and can be generally conceptualized as "thoughtful planning." An instructor might attend a workshop at a conference and adapt the workshop materials to design a new course. Two or three instructors at one or more institutions might work together to redesign a course or to design one "unit" in a course that they are all going to teach. And then again, an entire department could design a sequence of courses (e.g., the calculus sequence) that are designed to facilitate more student engagement.

The primary focus of this guide is designing for student-centered learning. This does not mean planning a lecture requires no design, but rather, designing a lecture traditionally has focused on writing notes and selecting examples to use during the lecture. Most instructors are comfortable with these practices of design. The purpose of this chapter is to define and illustrate the design work that can result in more active student engagement with full acknowledgement that this can be much harder and more time-consuming work. It requires more effort than writing notes for a lecture, but research shows such work makes a difference in student learning (CBMS, 2016; Freeman et al., 2014; NRC, 2015). Successful implementation of active engagement classroom components necessitates a shift in the instructor's role relative to traditional lecture that can result in students investing more time and effort in their learning. This extra effort has the potential to help students be more successful in their courses (Lee, Y., Rosenberg, J., Robinson, K., et al., 2016). The grounding principles of this design practices chapter are as follows.

Instruction should be designed for all, not just some, students. It is a common belief that individuals are either smart and able to do mathematics, or they are not. This can lead to an approach where teaching simply offers students an opportunity to see if they are in the "smart" category. There is also a tendency for instructors to believe that what worked for them as students will work in teaching others. Some might even claim that such teaching should continue in order to sustain the field of mathematics by producing mathematicians. The authors of this document do not subscribe to any of these beliefs. Instead, we consider mathematics to be a discipline to which all students should and can have access. The authors view teaching as a means to effect positive change in individual students and not as a means for sorting them.

Designing instruction is more than just planning content. Mathematics instruction cannot proceed under the assumption that the only important thing is to have students learn the mathematical content of a course. When designing instruction, it is important to consider developing mathematical practices, increasing access to the discipline, and encouraging positive dispositions toward mathematics.

Instructional design should aim to effect meaningful change. Mathematics instructors should be intentional in design with an aim toward instructional improvement. This intentionality includes articulating goals; envisioning how to help students achieve those goals through activities both in and out of the classroom; reflecting on the activities in terms of what content students learned, in what mathematical practices students are engaged, and what students' dispositions toward mathematics are; and revising for future use or revisiting lessons via additional activities that address the goals not yet achieved.

Design for student-centered learning must be in sync with evidence-based practices. Just as mathematics research forms the foundation of work in the field, teaching also has a foundation of research evidence. This chapter relies on that research and provides citations for further reading. Mathematicians need not become experts in the mathematics education research field to be successful teachers, but all instructors should be aware of evidence-based practices that can enhance student learning.

DP.1.1. Questions for design

The following is a list of questions we recommend instructors ask themselves at the beginning of a design process as a productive way to focus on intentional design.

1. Who are the students in this course? What knowledge and skills will students bring to this course?
    - Who am I in relation to my students, and how might that influence how they perceive me?
    - What kinds of variation will likely exist in my students' backgrounds?
    - What mathematical practices are my students in the habit of using?
    - What dispositions do my students have toward mathematics?
    - What do my students believe about who can learn mathematics?
2. What are the course learning goals?
    - What concepts and procedures do I want my students to master during this course? What does "understanding" mean in this course?
    - What will my students believe about mathematics as a discipline and as a way of knowing after this course? How will my students think about their relationship with mathematics after this course?
    - What are the central questions in this course, how will the course engage them, and why are these questions of value to my students?

- What goals do my students bring to the course? How will my students understand the course learning goals, and how will I help them understand the goals productively? What will I do to support my students in developing appropriate goals?
- How are the goals I articulate for this course related to those required or expected by the department or the institution? How can this course contribute to the larger educational goals for these students at this institution?
- What will convince me that a student has met the course goals?
- How will students know they are meeting course goals?

3. What does learning look like in the context of this course?
   - What experiences will allow my students to make progress toward the course learning goals, keeping in mind their current mathematical expertise and readiness?
   - How can my students experience competence, autonomy, or relatedness (as defined in section CP2.3) through this course?
   - How can tasks provide an appropriate level of cognitive demand? (See section CP.2.5 of the Classroom Practices chapter for a discussion of cognitive demand.)
   - Is the learning in this course supported with the kinds of reflection and metacognition needed to ensure the learning is deep and long-lasting?
4. What promotes student participation in the course?
   - To what extent are my students motivated? Are they motivated to engage deeply in course tasks or only superficially?
   - What is the impact of my students' motivations on their attitudes, beliefs, or behaviors?
   - Are my students participating at an appropriate level in authentic mathematical practices?
5. How is this course inclusive?
   - Does this course create barriers that disproportionately impact certain groups of my students? (E.g., will students who live off-campus have a harder time using support resources?) How will my students access and use available support resources?
   - How does this course manage the variation in student preparation levels, particularly at the beginning of the course or new units? Are there ways to use tasks in which my students can engage with "lower thresholds and higher ceilings" as described in the Classroom Practices chapter?
   - What is the plan for building an inclusive and equitable classroom as described in the following section of this chapter?
   - Is the course design flexible enough to adapt to specific learning needs? For example, how could a student with limited sight succeed in the course? Are there design choices that would naturally address needed accommodations? It is not necessary that every course be designed to address every possibility. Rather, the instructor should question foundational assumptions within the course design that may be false for some students.
6. How will I provide my students with feedback in this course?
   - How will I provide my students with formative feedback intended to help them make changes in the future?
   - How will I provide my students with summative feedback intended to help them assess how their work relates to my expectations?
   - How will my students understand and leverage this feedback?

7. How will I gather information to improve the course?
   - How will I use formative assessment to determine whether my students have met the learning goals?
   - How will I use summative assessment to improve the course?
   - Will I use mid-term course feedback forms or other tools? In what ways should this feedback be anonymous?
   - What questions will I ask on the end-of-term student feedback forms?
   - What unanswered questions do I currently have about the course design, and what data could inform changes to the course in this and future iterations?

In the Classroom Practices chapter are examples of paired board work, where it is easy for one student to do all the work without the other student(s) contributing. In an effort to ensure that all the students are engaged in the task at hand, it is useful for instructors to have different groups work on different tasks that are possibly more relevant to the students. For example, in an introductory statistics course it is helpful to have examples from the health sciences, the business world, the sports industry, and the environment because students from these various majors enroll in this course. Such personalised tasks can motivate students, engage them in meaningful mathematics, and help them to see the value of the tasks at hand.

Before moving forward we want to emphasize that the course syllabus is the primary venue for addressing some of the above questions, specifically topics centered on course structure, classroom protocols, behavioral expectations, learning outcomes, homework assignments, exams, and other assessments. The syllabus can help set expectations for an inclusive classroom climate and provide motivation and encouragement for students to engage deeply in the course tasks. It should also include information for students needing accommodations as detailed in section DP.2.7. Some institutions provide instructors with a syllabus template or checklist (e.g., academics.lmu.edu/media/lmuacademics/centerforteachingexcellence/ctedocuments/Syllabus%20Checklist.pdf), and books such as Grunert, Millis, and Cohen (2009) provide guidance.

DP.1.2. Considerations for design

Instructors need to be explicitly aware of many issues when designing instruction. This section comprises practical recommendations for evidence-based practices in design as documented in the research literature related to the following:

1. Equity
2. Learning goals for students
3. Research on supporting learning for all
4. Situational factors
5. Learning environments
6. Tasks and activities
7. Homework
8. Formative and summative assessment
9. Reflective instruction

DP.1.3. Designing for equity

Equity research plays a prominent role in the mathematics classroom. Instructor-focused pedagogies tend to ignore that students learn differently and treat all students equally, which can be inequitable, while student-centered pedagogies have the flexibility to support students in more individualized and more equitable ways.

Gutiérrez' (2009) Four Dimensions of Equity details four key aspects of the educational process that require attention:

- **Access**: This refers to the ability to gain intellectual and physical access to mathematical ideas and mathematical teaching and learning spaces (e.g., classrooms, tutoring centers, office hours, and informal interactions).
- **Achievement**: This refers to students' success in mathematics as traditionally measured (e.g., performing well on homework and exams, succeeding in courses, and majoring in fields requiring mathematical knowledge).
- **Identity**: This refers to who our students are, including the resources and ways of knowing they bring to the learning environment, and to who they become through their participation in mathematics.
- **Power**: This refers to attending to the distribution of power between instructor and student, between students, and between students and mathematics (e.g., constructor of knowledge versus passive receiver of knowledge, mathematics as an empowering force versus mathematics as a barrier).

We will examine each of these aspects more thoroughly through the lens of course design, and we will pose questions related to equity upon which a course instructor or developer may reflect. See section XE.2.1 in the Equity section of the Cross-cutting Themes chapter for more information.

Access

Does the nature of this course set up barriers that disproportionately impact certain groups of students (e.g., multilingual students, historically marginalized students of color, women, students with disabilities)?

- *How can I group my students in equitable ways?* For example, how does the nature of a given mathematical task lend itself toward certain forms of student grouping (e.g., heterogeneous, mixed-ability pairs versus homogeneous groups of three or four)? How might I structure group problem-solving opportunities in the classroom so that the mathematics is accessible to all my students?
- *How can I approach course design in ways that promote mathematics as a discipline in which all my students feel they belong and can grow?* For example, are my course activities designed with multiple entry points that can accommodate students with diverse learning needs (e.g., students with disabilities, multilingual students)? Do my students with different life circumstances (e.g., live off-campus, have a full-time job) have equitable access to course resources?

Achievement

How do the participation structures and assessments in this course allow all students to demonstrate their understanding and inform all students how to advance their learning? Do assessment results provide meaningful data about disparities in learning outcomes?

- *How will my students participate meaningfully in mathematical work in this course?* For example, tasks with many entry points ("lower floors") and with potential for extensions and connection to more complex mathematics ("higher ceilings") can be used to engage all students in the mathematical work. Participation structures such as think-pair-share, small group explorations, and student presentations can also be used to create more opportunities for students to participate, contribute meaningfully in class, and assess their own understanding.
- *How will assessments provide meaningful feedback to my students about their learning and provide insight for the instructor into ways the course can address all students' needs?* Results of assessments provide critical information to students about what they have learned and what additional work they must

do to be successful in the course and beyond. Instructors can empower students by helping them learn how to prepare for assessments, anticipate what will be assessed, and interpret and respond to assessment results. Assessment can also reveal for whom this course is working and for whom it is not. For example, poor performance by English language learners may indicate that either the course activities or the assessments should be redesigned.

Identity

In what ways does this course design recognize students' membership and positioning in society and work toward the development of positive social and mathematical identities?

- *How does my course design acknowledge and affirm my students' social identities in learning mathematics?* For example, in what ways are affective aspects of mathematical problem solving (e.g., perseverance, learning from constructive critiques of reasoning) valued and taken into account in the course learning outcomes and assessment of those learning outcomes? How is classroom participation in the course structured in ways that mitigate traditional stereotype threat and implicit biases related to gender, race/ethnicity, other types of status related to mathematical ability? For example, group work opportunities and sharing samples of student work can disrupt preconceived ideas of peers' competence and skills.
- *To what extent do I value knowledge and experiences that my students bring to the course as resources for mathematics learning?* For example, how can my course activities and assessments leverage my students' social backgrounds while they learn mathematics? In what ways does the course design allow my students to demonstrate and engage in different forms of agency as mathematics learners? Examples of this include using mathematics to challenge the status quo and providing space for different types of contributions in mathematical problem solving and reasoning.

Power

How does this course support students in constructing mathematical knowledge and empower students through mathematics?

- *How will I enable my students to access and take ownership of mathematical ideas?* Research on learning suggests that in order to construct knowledge, people must engage in activities that promote intellectual stimulation and growth such as asking questions, posing problems, and making mistakes. How do key course activities support my students in the intellectual "heavy lifting" required for learning? How does the course design promote a culture of positivity, encourage mistake making, and support risk taking?
- *How will I balance my students' ways of knowing with traditional mathematics content and practices?* Instructors should create a safe environment in which students feel comfortable bringing their ways of thinking about, engaging with, and making sense of mathematics to the classroom. The instructor must empower students to view mathematical ideas as having validity based on the structure of mathematics itself rather than an instructor's decision about validity. At the same time, the instructor is responsible for conveying to students the traditions, usage, and conventions of mathematics as a discipline and thereby empowering students as members of the mathematical sciences community.
- *How does my course empower my students through mathematics?* How does the course help my students recognize the presence and utility of mathematics in their lives? For example, students in calculus could be asked to discuss with other professors in their major how a problem showcases the utility of the content they are learning in their mathematics class. How can my course help my students make sense of critical issues in the world around them through mathematics?

The following illustration typifies an approach to the design of an activity that uses a low-floor, high-ceiling task to establish a classroom community with mathematical power shared by students and the instructor. This illustration is written from the perspective of an instructor three weeks into a liberal arts mathematics course planning a lesson for the following week.

Classroom vignette: Designing a lesson for a liberal arts mathematics class

At the beginning of the course Professor Evans provides each of the thirty freshmen students with a Rubik's cube, and they work hard on solving the cube for the first three weeks of the semester. Giving each student a cube alleviates issues for students who either cannot afford the cube or cannot check them out at the reference desk of the library because they live off-campus.

The students work on the cube in class on Thursdays, and the professor covers other course topics on Tuesdays. After most students can solve the first layer of the cube, Professor Evans considers the course goals and the following issues:

1. Some students are still making sense of how the cube works and do not yet see how corner cubies are different from edge cubies.
2. There are two sets of students: those who have completed the first layer and those who have not. The instructor uses ideas discussed in the Classroom Practices chapter to determine how to assign students in each set to groups.
3. Students who have completed the first layer are grouped to work on the next task: making sense of a powerful Rubik's cube technique called "M1". They will use representations other than verbal descriptions and Singmaster notation (used in Rubik's Cube solution guides) to communicate sequences of moves. The goal is for students to become more flexible in reasoning about longer sequences of moves frequently used to effect position changes of corner cubies.
4. Students who are still working to solve the first layer begin working together in groups as well. Students who solve the first layer will move to one of the groups working on the M1 task. If several students solve the first layer at about the same time, they form an M1 group of their own.

An important part of this activity is the way that students communicate about sequences of moves, since efficient representation of changes to the cube can aid in the solution process. In order to allow students to engage in productive struggle, the professor resists the urge to help too much and instead encourages them to talk with other groups and compare representations. The professor asks groups to explain their representations to the class early in the solution process so the class recognizes the power of notation to aid in a solution.

The assessment is complete when all the students have demonstrated their solution to the first layer. All students will earn 100% because this learning opportunity is designed to be accessible to all students. The instructor acknowledges that it is impossbile to plan what will happen in every minute of the class because so much depends on what the students do, so instead a rough outline is developed.

DP.2. Student learning outcomes and instructional design

There are a variety of terms (e.g., learning outcomes, learning goals, learning objectives, competencies, student benchmarks) used to describe what students should know upon exiting a lesson or course. Student learning outcomes can include **content, cognitive**, and **affective** goals (e.g., MAA, 2015; CCSSM, 2010) and are often tied to pedagogical goals or teaching practices (e.g., Blair, 2006; NCTM, 2015). Content goals are explicit skills and understandings often central to the design of a course. Cognitive goals are less course-specific and include understandings about the practice of mathematics. Affective goals pertain to elements of students' learning that relate to their dispositions toward and emotions about mathematics.

Content, cognitive, and affective goals are important features of successful instructional design and go well beyond traditional assessment measures. Instructors should identify student learning outcomes that are robust and practical enough to guide instructional design, and instructors should design instruction that effectively supports specific student learning outcomes.

Content goals

The CUPM guide (MAA, 2015) lists four cognitive recommendations, nine core content recommendations for programs in the mathematical sciences, and content goals for a variety of specific courses as well.

CUPM Guide: Content recommendations (MAA, 2015)

1. Mathematical sciences major programs should include concepts and methods from calculus and linear algebra.
2. Students majoring in the mathematical sciences should learn to read, understand, analyze, and produce proofs at increasing depth as they progress through a major.
3. Mathematical sciences major programs should include concepts and methods from data analysis, computing, and mathematical modeling.
4. Mathematical sciences major programs should present key ideas and concepts from a variety of perspectives to demonstrate the breadth of mathematics.
5. Students majoring in the mathematical sciences should experience mathematics from the perspective of another discipline.
6. Mathematical sciences major programs should present key ideas from complementary points of view: continuous and discrete; algebraic and geometric; deterministic and stochastic; exact and approximate.
7. Mathematical sciences major programs should require the study of at least one mathematical area in depth, with a sequence of upper-level courses.
8. Students majoring in the mathematical sciences should work, independently or in a small group, on a substantial mathematical project that involves techniques and concepts beyond the typical content of a single course.
9. Mathematical sciences major programs should offer their students an orientation to careers in mathematics.

In addition, the Assessment Practices chapter of this guide provides advice about specifying measurable content goals and specific learning outcomes to enable effective assessment. For the purposes of the Design Practices chapter, designing activities to meet content goals through student engagement involves deliberate selection of tasks that focus on the mathematical content specified in the goal. Content goals then serve an important framework for the design of the rest of the course.

We offer two illustrations of designing content goals. The first demonstrates designing a class activity to meet a student learning outcome and the second demonstrates using content goals at the course level.

Classroom vignette: Designing an activity for a college algebra class

Student learning outcome (content goal): Describe the meaning of an algebraic formula.

Assessment: Write a paragraph describing the connections between an algebraic formula and a formula intended to describe a computation.

Activity: *Fast Food Efficiency.* Each evening, shift managers at a fast food restaurant have to enter data into a form and perform a calculation. This "French fries efficiency" calculation indicates what percentage of fries cooked in a given day were actually sold as opposed to wasted, e.g., dropped on the ground, thrown out, or eaten by employees.

Students simulate the manager's task by taking four samples of data on four separate days and completing the form. Students then encode variables for each non-constant quantity, connect each step using an appropriate operation, write the formula, and program the formula into a spreadsheet to perform calculations for each day of an entire month.

Classroom vignette: Designing a hybrid quantitative reasoning course

The college of business contacted us requesting that we design a sequence of courses for their majors. At that time students were required to complete an intermediate algebra course to fulfill the university mathematics requirement. Over the course of several meetings, the business faculty identified necessary student proficiencies in quantitative reasoning, problem solving, and algebra content. A mathematics instructor offered to take the lead in developing two courses: Quantitative Reasoning for Professionals 1 and 2. This instructor determined appropriate student learning outcomes for each course and broke each outcome into several objectives which formed the basis for individual lessons and assessments.

Quantitative Reasoning for Professionals 1: Student learning outcomes

Students should:

1. Apply prior knowledge and mathematical concepts to solve novel problems.
2. Use proportional reasoning to solve problems.
3. Use data to make and defend decisions.
4. Construct algebraic formulas to model real-world quantitative relationships.
5. Manipulate formulas involving a variety of mathematical operations.

Quantitative Reasoning for Professionals 2: Student learning outcomes

Students should:

1. Apply prior knowledge and mathematical concepts to solve complicated, novel problems in context.
2. Identify and create models of linear functions involving verbal, numerical, algebraic, and graphical representations.
3. Identify and create models of exponential functions involving verbal, numerical, algebraic, and graphical representations.
4. Solve problems requiring the use of logarithms.
5. Use linear systems of equations and inequalities as well as linear programming to solve problems.

For each course, the first outcome focuses on problem solving rather than content, and the remaining outcomes are content based. The second course increases the expectations for problem solving beyond those in the first course. In order to support problem solving, we often schedule these classes in rooms that support group work, and we utilize a form of inquiry-based learning to provide a meaningful experience for the students. Some classes have been linked with English composition classes in order to promote communication skills development in connection with problem solving.

Cognitive goals

The CUPM guide (MAA, 2015) and *Beyond Crossroads* (Blair, 2006) provide starting points for instructors to identify or create cognitive goals that are robust and practical for guiding instructional design. These reports focus on cognitive goals that support the development of mathematical habits of mind (MAA, 2015, p. 10). More specifically, among the professional associations focusing on undergraduate mathematics education, there is consensus that students should learn how to communicate mathematics effectively, develop independence in problem solving, and learn how to work with technology (Saxe and Braddy, 2016).

Once instructors establish cognitive goals they can design instruction to support these goals. For exam-

ple, if an instructor sets a cognitive goal for developing effective mathematical communication skills, the course activities should engage students in mathematical communication. The instructor should consider a variety of ways that students should be able to communicate mathematical ideas, and then choose instructional strategies that require students to speak, write, read, compare, critique, and compose mathematical explanations.
The CUPM guide (MAA, 2015, pp. 10–13) delineates four cognitive goals for programs in the mathematical sciences.

CUPM Guide: Examples of cognitive goals (MAA, 2015)

1. Students should develop effective thinking and communication skills.
2. Students should learn to link applications and theory.
3. Students should learn to use technological tools.
4. Students should develop mathematical independence and experience open-ended inquiry.

Affective goals

Most instructors have the least experience with student learning outcomes that fall into the category of affective goals. "Affect is a disposition or tendency or an emotion or feeling attached to an idea or object. Affect is [composed] of emotions, attitudes, and beliefs" (Phillip, 2007, p. 259). Affective goals complement content and cognitive goals. Goals that address student access to learning, power, identity, confidence, enjoyment, creativity, curiosity, ability to work with others on mathematical tasks, and ability to seek help and accept and respond to feedback are all examples of affective factors.

Setting affective goals for a course is especially important in light of increasing research evidence that the way in which instruction is designed and delivered can directly affect the motivation, confidence, engagement, curiosity, persistence, and other affective factors for a growing number of underachieving or underrepresented students (Ellis, Fosdick, and Rasmussen, 2016). In turn, "positive self-perceptions such as increased perseverance, risk-taking, and the use of improved cognitive and self-regulatory strategies are connected to efficient and deep learning" (Hassi and Laursen, 2015, p. 319). Furthermore, designing and implementing student-centered instruction has been shown to improve certain affective factors for students of color and first-generation college students (e.g., Kelly and Hogan, 2014).

Affective factors are not typically measured or graded via traditional assessment methods. In a student-centered classroom, instructors can monitor many of these factors via formative assessment and make instructional design decisions based on student progress. Instructors should be purposeful in selecting classroom activities that provide opportunities to achieve these affective goals and assessments that provide insight into student progress.

Here is an example of an affective goal used for design:
Affective goal: Students should perceive mathematics as a discipline in which they can participate and to which they can contribute.

Assessment: On the final exam the instructor asks students to reflect on the following prompt, "Describe one way in which you have participated in the discipline of mathematics through this class." The instructor reads these reflections to understand how topics in the class and structure of the class are interpreted by the students. In future iterations the instructor modifies the course, if necessary, to alleviate some of the logistical and structural barriers.

The next illustration demonstrates how an instructor uses student learning outcomes to redesign a course.

Classroom vignette: Redesigning a capstone course for mathematics majors

Professor Baker was given the task of redesigning a senior capstone course for mathematics majors. The course is offered annually to about 20 graduating mathematics majors, more than half of whom are pursuing teaching careers. Prior instructors and students commented that the projects and course content needed to be revised. Professor Baker's course redesign involved working with a group of instructors for a week during the summer as a course design community of practice and using evidence-based design practices to redesign the course. Their goals were to rethink the student learning outcomes and evaluation criteria and develop a new project assignment focused on students building and demonstrating problem solving and critical thinking skills. Edwards and Hamson (2007) and resources developed by CoMAP (www.comap.com/) helped to inform the selection of the content.

Step 1: Rethinking student learning outcomes

Rewriting the student learning outcomes helped clarify the most important aspects of the course. The comparison table below shows the original and redesigned learning outcomes. Changes included adding verbs from Bloom's Taxonomy (www.bloomstaxonomy.org/) to indicate the level of student reasoning expected. The team of instructors also added new learning outcomes to better represent the goals for the course within the department's mathematics program.

Original learning outcomes	Redesigned learning outcomes
1. Understand the basic mathematical modeling process. 2. Solve typical mathematical modeling problems using known models of two types: Monte Carlo simulations and differential equations models. 3. Analyze the solution of mathematical modeling problems. 4. Write reports and make oral presentations on the results of modeling projects.	1. Apply basic mathematical modeling strategies to solve typical application problems in the physical, social, life, information, and engineering sciences. 2. Effectively analyze and evaluate the quality of mathematical models and model-based interpretations. 3. Find and synthesize connections within and across secondary and postsecondary mathematics content. 4. Use modern computing software as a tool for visualization, simulation, and analysis of mathematical models. 5. Effectively communicate mathematical modeling processes and outcomes in both written and oral forms.

Step 2: Linking assessment to student learning outcomes

The team mapped the major course assessments, learning outcomes, and relative weights in the grading scheme in the syllabus so that the students would see a clear connection between the student learning goals and their earned grades.

Assessments	Learning outcomes	Weight
Homework	1, 2, 3, 4, 5	10%
Project #1: Pose a modeling problem	3, 5	15%
Project #2: Solve a modeling problem	1, 2, 4, 5	30%
Major field test	3	10%
Skills mastery quizzes	1, 3, 4	25%
Final presentation	2, 5	10%

Step 3: Creating guidelines for student projects

Professor Baker structured the semester-long modeling projects to be completed by teams of three to four students using the following four-step process: (1) researching a modeling application in a small group, (2) developing a problem scenario that might be reasonably solved by another group in the class, (3) solving another group's problem scenario from the prior step, and (4) writing and presenting a technical report of findings.

Project reports required background, problem description, solutions, implications, and limitations sections and drew on examples from the course textbook. The norms for group collaboration were already strong among students in the class, but Professor Baker clearly articulated the expectations for teamwork as an added layer of accountability.

To assess student work the team developed rubrics for the projects that included guidelines for peer review of presentations and instructor evaluation of presentations and used the rubrics to inform students of our expectations up front. The rubrics followed the simplified step-down analytic format (Bean, 2011) and utilized American Association of Colleges and Universities frameworks to clarify the meaning of critical reasoning and problem solving (www.aacu.org/summerinstitutes/igea/curriculum).

DP.2.1. Designing the learning environment

There are factors specific to each class that influence the learning environment and an instructor's day-to-day work. Examples include the differences in students' mathematical backgrounds, the number and length of class meetings (e.g., 50 minute versus 75 minute periods), and the purpose of the course (e.g., it serves as a prerequisite for another course). An instructor cannot assume that a student who just finished high school with AP credit will engage in a calculus course the same way as a student taking the course after completing precalculus as an undergraduate.

The way students engage in a course is very different depending on whether the course is online, meets face-to-face, or has been flipped (see "Designing a flipped classroom" section below). The number of students in a class influences activity design choices. For example, in a small class the instructor might require students to make presentations, but this might not be as feasible in a large class. If a class is flipped and all lectures are delivered via online videos, then the instructor must plan more activities for class time.

A factor over which instructors may have little control is the physical environment. Instructors must work within the constraints of institutional resources. This includes the size of the classroom, the configuration of the room, and the availability of technology. On the other hand, instructors have significant control over some "soft" aspects of a learning environment regardless of the physical classroom setting. For example, an instructor might have the students move their desks together in order to facilitate small group work or might assign seats to ensure students are not sitting only with students who are similar to them.

Note that research on learning environments can be useful to lobby for institutional investment in smaller class sizes, classrooms conducive to active student engagement, and up-to-date technology. An example of a physical classroom environment developed to facilitate increased student engagement exists in the field of physics. *Scale-up* classrooms offer students the opportunity to work in small groups, large groups, and as a class using computers and white boards for groups, as needed (scaleup.ncsu.edu/FAQs.html).

The learning environment can support or constrain the intellectual space. For example, it is important to create a place where students feel comfortable communicating with their instructor and peers, feel comfortable making mistakes, and feel they are making valuable contributions to the class. An environment that encourages students to take responsibility for their own learning might require the instructor not always be in the front of the classroom. It might involve students working at the board or in small groups using individual whiteboards. It might involve students presenting ideas in small groups, to the entire class, or to the instructor one-on-one. These ideas along with supporting evidence are discussed further in the Classroom Practices chapter.

DP.2.2. Designing mathematical activities and interactive discussions

For most instructors designing the actual tasks and activities for a course is the most important part of the design process, and this aspect of the process is addressed more extensively in the Classroom Practices chapter. In the present chapter we briefly discuss the development of mathematical tasks that allow students to engage with the mathematics in meaningful ways. The tasks should be engaging and interesting to students and have high-level cognitive demand. (See section CP.2.5 of the Classroom Practices chapter for a discussion of cognitive demand.) Instructors must carefully consider how to launch complex tasks in order to ensure they are accessible to all students.

We design instruction not only to introduce new content but to review previous content as well. Sometimes this involves brief mini-lectures followed by interactive discussions. At other times an instructor will design for the introduction of formal mathematical language such as the definition of a mathematical object or the statement of a theorem. Two schools of thought exist based on how mathematicians operate, and the design choice depends on the pedagogical goals of the lesson. Sometimes it is helpful to include formal language with the introduction of new content in order for students to begin working through ideas using appropriate language. On the other hand, it may be advantageous to allow students to develop new mathematical ideas first, then introduce the language to describe the mathematics. When the goal is for students to reinvent the mathematics to better understand the ideas, instructors must determine how long to wait before summarizing students' reinventions and guiding them to formal definitions and theorem statements. On the other hand, if the goal is to demonstrate the progression of mathematical idea development, using formal language from the beginning may be helpful.

The following vignette focuses on designing an interactive activity and discussion.

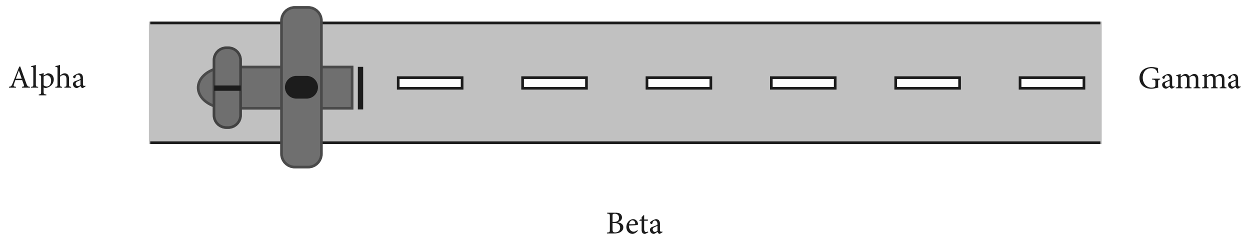
Classroom vignette: Designing a content unit for a differential equations course

One of the ten units in a differential equations course focuses on introducing solutions to systems of differential equations in a setting in which students worked in groups. The affective goals focus on empowering students to develop the mathematics themselves and take responsibility for their own learning. The cognitive goals include students learning to work collaboratively, to communicate effectively both verbally and in writing, and to visualize as a way to understand mathematics. The learning outcomes focus on developing student notions of a three-dimensional solution to a system of differential equations, guiding students to recreate Euler's method for systems, and developing student understanding of a phase plane representation of a system.

The instructor begins by creating an activity where students visualize a solution.

Three dimensional visualizations. A crop duster plane with a two-blade propeller is rolling down a runway. On the end of one of the propeller blades, which are rotating clockwise at a slow constant speed, is a noticeable red paint mark. Imagine that for the first several rotations of the propeller blades the red mark leaves a "trace" in the air as the plane makes its way down the runway.

Simulate this scenario over time with your finger or create the trace with a pipe cleaner. Sketch the ideal perspective (what you would see) for persons at alpha, beta, gamma, and delta. What view do you think is the best and why?



(Person Delta is located in a hot air balloon directly above the airplane.)

The content goal for this task is straightforward. Students visualize a three-dimensional curve and reason about its properties. The unit continues with tasks whose content goals are that students learn to use Euler's method in three dimensions and to develop a phase plane. This unit was designed to help students to reason at a big-picture level about solutions to systems of differential equations and not just to memorize procedures to solve them. In a later unit students develop procedures to solve linear systems of differential equations using the following activity:

> A group of scientists wants to graphically display the predications for many different nonnegative initial conditions to the rabbit-fox system of differential equations and they want to do so using only one set of axes. What single set of axes would you recommend they use (*R*-*F*-*t* axes, *t*-*R* axes, *t*-*F* axes, or *R*-*F* axes)? Explain.

Assessment of student learning after the completion of these tasks involves students writing about their understanding in homework, quizzes, or exams. The early development of this unit included reflection on the results and multiple revisions.

DP.2.3. Designing homework

There are differences between "assigning homework" and "designing homework." When designing homework, the intentionality of the learning outcomes must be considered. It may not always be appropriate to simply assign a list of problems out of a textbook. Instructors should be cognizant of the purpose for each assignment and be deliberate in their choices.

Sometimes homework is about skill building. In such cases assigning a set of textbook problems may be a sound choice. However, assigning homework may be related to other goals such as building students' understanding and conceptualizations of mathematical ideas or helping students reflect on what happened in class. In these cases homework might consist of writing a reflection, watching a video and responding to a prompt, emailing the instructor an answer to a conceptual question, or emailing the instructor a list of questions regarding a particular concept.

What is important and feasible in one course may not be the same in other courses. For example, an instructor's goal for the homework may be for students to think ahead about new material, as in the case of Dr. Gomez in the second classroom vignette of section CP.1.6 of the Classroom Practices chapter. This goal might be important in a course where one of the instructor's goals for the students is to learn to read mathematics for understanding, but it may be impractical in another course.

Instructors may design homework that encourages student collaboration and mutual support for their peers. For example, it might take as little as five minutes during a designed class activity for students to connect with peers with whom they will study outside of class. Homework assignments may be created when a unit is designed or after a class has occurred. For example, in a student-centered classroom it can be difficult to know in advance what students will discuss and which concepts they might struggle to understand. If the homework assignments are completely detailed in the syllabus, there is no room for flexibility.

DP.2.4 Designing a flipped classroom

The notion of a flipped classroom is discussed in the Classroom Practices chapter. In this chapter we discuss design practices that need to be considered when flipping a classroom. In instructional designing it can be challenging to manage the allotted class time to give students the best opportunity for success. One strategy that provides a great deal of flexibility in how class time is used is known as "flipping the classroom." In this pedagogical method the instructor gives students the responsibility to complete readings or watch short video lectures prior to class—that is, the "instruction" is done outside of class. Students then come to class having already been introduced to content which allows the instructor flexibility in engaging students during class time. For example, an instructor can ask students to bring questions that arise from their reading or from watching the video and spend some class time discussing those questions. If class time is freed of content delivery, time can be spent with students engaged in more collaborative activities.

An instructor who decides to implement the flipped classroom model must make informed decisions about how students will be engaged outside of class and how these outside activities will relate to those in class. For example, an instructor may decide either to find video or to create a video explaining a mathematical concept and then provide examples of how to work with this concept. During class time the instructor can have students work in small groups to solve a few carefully chosen problems that push students to reason critically about applying this concept in different situations. For additional information and examples, see the Classroom Practices chapter.

DP.2.5 Using formative and summative assessment in design

Formative and summative assessment are both defined in the Assessment Practices Chapter but they also need to be considered as part of instructional design. **Formative assessment** can be student activities that allow students to better understand what they are learning and to assess their own progress in learning. It can also be an activity that instructors use to inform their own instruction. In every lesson the instructor should include opportunities to gauge student progress as students proceed through the content. This allows the instructor to modify instructional plans to better accommodate students. For example, in student-centered classrooms the instructor receives a continuous stream of formative assessment data as students grapple with their assigned tasks.

Another example of formative assessment is the use of brief closure activities. The closure activity could be an exit ticket (as described in section CP.1.4 of the Classroom Practices chapter), such as an index card on which students respond to questions such as, "What is the clearest and what is the muddiest point from class today?" Another closure activity (also described in section CP.1.4 of the Classroom Practices chapter) is the "one-minute paper" in which students spend one minute summarizing the class period in writing. The instructor can use responses to such prompts to modify the following day's plans in order to better address students' questions, concerns, and misunderstandings.

Summative assessment is used to evaluate student learning and can occur in myriad ways, often with the idea that the class is moving on to a new topic. Examples include an end of unit exam, an end of course exam, and a portfolio project. More information about summative assessment can be found in the Assessment Practices chapter.

DP.2.6. Reflective instruction

Explicit reflection on the implementation of a lesson is an important component of design. Good design practices include reflecting on how things went during and after individual lessons and after the course as a whole. Some questions that can guide reflection are: Did the students participate as I hoped? Were the learning outcomes met? Did the environment support conceptual learning? What went well and why? What needs improvement and how might it be improved? Are there student comments I want to remember to help recreate the same positive outcome in the future?

Instructors have all had days where they think, "Wow! I hope I can do that again," or "I need to do that differently next time." Such real-time reflection may get lost if not recorded. For example, if an activity was not accessible to all students, the instructor should note it and make revisions for the next time. Reflection is also an opportunity to maintain alignment of outcomes, learning activities, and assessments. Reflection time at the end of instruction is essential in order to ensure learning goals are met, but this can be challenging for instructors who teach back-to-back classes or have other impediments to thorough reflection.

DP.2.7. Students needing accommodations

As mathematicians work toward designing courses that incorporate active engagement strategies and provide inclusive learning environments, they must attend to the needs of students with disabilities. In 2011–12, 11% of undergraduate students self-identified as having a disability (National Center for Education Statistics, 2016). These students reported that they had one or more of the following conditions: a specific learning disability (e.g., ADHD, dyslexia, dysgraphia, mathematics anxiety), or a visual, hearing, speech, orthopedic, or health impairment. Students may also have a diagnosis of autism spectrum disorder which encompasses four separate disorders: autistic disorder, Asperger's disorder, childhood disintegrative disorder, and pervasive developmental disorder not otherwise specified.

According to the Americans with Disabilities Act (ADA) of 1990, higher education institutions are responsible for making reasonable accommodations when a student provides documentation of a disability (www.apa.org/pi/disability/dart/legal/ada-basics.aspx). The purpose of any accommodation is to give the student an equal opportunity to participate in an academic program. Depending on the need this may involve a note taker, extra time on exams, a quiet testing environment, large-print materials, image-enhancing technology, audio-recorded materials, flexible due dates (e.g., for a student with a temporary disability like pregnancy), or wheelchair friendly furniture. See www.apa.org/pi/disability/dart/toolkit-three.aspx for a more comprehensive list.

To request an accommodation a student will typically apply to a designated campus office that provides support for students with disabilities. After the office determines whether the student is eligible for accommodations it will communicate and coordinate with instructors, student housing staff, and other departments on behalf of the student. Having students submit requests for accommodations to a designated office ensures consistency across the institution and removes from the instructor the burden of determining whether an accommodation is appropriate. All course syllabi should detail the process students with special needs are to follow to request reasonable modifications, special assistance, or accommodations in a course.

Students need not disclose their specific disability to instructors, and it is a privacy infringement to request such information. The campus disability services office can be a resource for instructors teaching students with disabilities. For examples of the types of resources and information supplied by disability services offices, see Texas A&M University (disability.tamu.edu/facultyguide/teaching), Towson University (www.towson.edu/dss/dss-faculty-guide-2015.pdf), and Vanderbilt University (cft.vanderbilt.edu/guides-sub-pages/disabilities).

While the MAA Instructional Practices Guide does not discuss teaching mathematics to students with disabilities in detail, the following may be of interest. For information on teaching students with Asperger's syndrome see Langford-Von Glahn, Zakrajsek, and Pletcher-Rood (2008). Sullivan (2005) wrote about her experience teaching three students with learning disabilities in a general education mathematics course that emphasized making sense of mathematics and engaging in mathematical discourse. Jackson (2002) revealed the falsity of many stereotypes about blind mathematicians. As instructors include more active engagement techniques in their teaching, they should be mindful of both how activities may be inaccessible for students with disabilities and what instructors can do to provide all students with an equal opportunity to participate in the activities. For example, how can the instructor support a note taker or sign language interpreter during group work? How will manipulatives need to modified for a blind student? What supports might benefit a student with Asperger's syndrome during social interactions such as group work, think-pair-share, and class discussions? The answers to such questions are highly dependent on individual students and should be navigated on a case-by-case basis.

DP.3. Challenges and opportunities

Successful implementation of engaging classroom components (a) necessitates a shift in the instructor's role relative to traditional lecture, and (b) tends to alter the time and effort cost for students (Lee, Y., Rosenberg, J., Robinson, K., et al., 2016). In this section we discuss challenges and opportunities related to designing an active engagement environment.

DP.3.1. Big-picture challenges and opportunities

There are two broad challenges with active-engagement design, and with these challenges come associated opportunities. The first challenge is designing instruction and activities that depend on prerequisite knowledge and skills. According to Megginson (n.d.), lessons designed requiring advance preparation by students (e.g., reading the textbook, watching videos) stand in stark contrast to lessons designed with no preparation expected from the students. The associated opportunity is designing a lesson that extends, clarifies, and enriches students' knowledge rather than a lesson introducing content for the first time or reintroducing content that students already have experienced. From a design perspective a lesson that meets this challenge and capitalizes on this opportunity should center on questions related to the advance preparation activities as well as have entry points to the in-class portion for students who have not appropriately prepared.

The second challenge arises from the fact that designing lessons with active engagement does not guarantee that learning will take place. An active environment without the proper tasks, quality student interactions, or timely feedback may actually undermine, not amplify, the learning experience. This provides the associated opportunity to design a lesson that leverages real-time, student-driven problem solving towards increased conceptual engagement in and ownership of the mathematics. For example, the use of clickers or whiteboard peer-collaborations are evidence-based pedagogies with the potential to have a positive impact on student learning (Mulnix, A. B., Vandegrift, E. V., and Chaudhury, S. R, 2016; National Research Council, 2015), but it is ultimately the instructor's management of the active engagement tasks that realizes these opportunities. Thus, from a design perspective, in order to maximize learning, instructors must utilize a variety of strategies to alleviate struggles and misconceptions that influence real-time student work toward a deeper understanding of content and connections (e.g., CCSSM, 2010; McCallum, 2015; Schmidt, W. H., McKnight, C. C., and Raizen, S. A., 1997).

DP.3.2. Other challenges

Each of the following challenges in course design are explored in more depth below.

1. **Time to prepare.** It initially takes longer to prepare student engagement activities than to prepare lectures.
2. **Judging effectiveness.** Instructors should engage in a continuous cycle of reflection and revision rather than reusing the same materials without modifying them.
3. **Achievement.** Instructors may need to recalibrate assessments to reflect updated expectations of students' work.

4. **Content coverage.** Instructors may have to reduce the content covered which will require judicious decisions about what to remove.
5. **Buy-in.** Instructors may need to manage expectations that students and colleagues have based on their prior experiences and beliefs.

Preparation time

When incorporating changes to fundamental pedagogy an instructor can expect an increased time commitment in designing or redesigning a course. For example, if an instructor designs a flipped classroom, they may invest considerable time creating videos for students to watch prior to class, corresponding activities, and in-class activities. Of the 1089 instructors who responded to a survey of *Faculty Focus* readers, the majority found time limitations to be a significant hurdle to flipping a classroom. "Although lack of support was clearly a limiting factor, the biggest barrier of all was time. Approximately 38% of survey participants indicated that time was 'always a challenge' and another 31.61% said time was 'often a challenge'" (www.facultyfocus.com/articles/blended-flipped-learning/flipped-classroom-survey-highlights-benefits-and-challenges/). If a course is taught repeatedly using a similar approach and an instructor continues to refine their methods, the addtional time commitment will naturally decrease and stabilize.

Judging effectiveness

Instructors engaged in design practices should expect periods of adjustment and should withhold judgement about the effectiveness of a new approach until they have tried and modified the approach multiple times over a period of weeks or successive terms. During the initial implementation of a particular design, frequent written feedback from students can inform improvements that might be made. Negative student feedback might tempt an instructor to revert to a lecture-based class, but the instructor should remember that no single approach works for all classes or all lessons and should allow time to adjust the design each successive term.

While instructors should make adjustments at the end of each term, adjustments might also be required in real time. An instructor must understand that there will be moments when lessons move in unanticipated directions due to novel student ideas or questions and must evaluate the potential cost and benefit to leverage the new learning opportunity (Herbst, 2008). The instructor can facilitate such an opportunity by posing key questions that move the discussion in a productive direction.

Departments should also withhold early judgement and support instructors as they employ new approaches. Student evaluations may suffer during early implementation of a student-centered design. For example, students may complain that an instructor just stands there and makes them do all the work. Untenured and contingent instructors particularly require support from colleagues and chairs as they work toward becoming more effective instructors. In the end, students, instructors, and administrators should all be most concerned with knowledge and understanding attained by students regardless of which instructional approach is used.

Achievement

Most instructors have learned to calibrate assessments to their student learning outcomes. Using novel instructional approaches can result in a shift in student achievement. For example, students who are accustomed to performing well on procedural tasks may not do as well on conceptual tasks, and as a result their grades might suffer. On the other hand, students who may struggle with procedural skills may be pleasantly surprised to find they are more successful with conceptual tasks. Such shifts can create tension in a class or a department. The design process should build in appropriate student supports to mitigate the effects of novel instructional approaches and the corresponding changes in some students' grades.

Content coverage

One of the biggest concerns when redesigning a course is ensuring all the necessary content is covered. Instructors may need to reconsider how class time is used in order to achieve the student learning outcomes for the course, especially when the course serves as a prerequisite or covers standardized content for various stakeholders. An instructor should expect that content may not be covered at the same pace as before the redesign, as students may learn some topics faster with a new approach while others take longer. Knowing that students arrive with different levels of preparation, an instructor should design the course with both underprepared and well-prepared students in mind. For example, the instructor might plan to assign additional work outside of class for underprepared students. To assist with content coverage an instructor may choose to incorporate instructional technologies that facilitate learning both in and out of the classroom. See the technology section in the Cross-cutting Themes chapter for more information.

Buy-in

In order to successfully implement a course design, instructors must have the support of both administrators and students when employing a new instructional approach. Instructors should set realistic expectations and create an atmosphere where students can be successful

Administrators should support instructors in identifying resources such as computers, tablets, manipulatives, video equipment, and even suitable classroom space. An appropriately sized classroom with furniture conducive to student engagement can be instrumental in the success of a student-centered course design.

Most students will expect and be comfortable in a traditional mathematics classroom environment. They understand the traditional implicit contract between students and instructor: the instructor delivers clear lectures with appropriate examples, and the students listen to the lecture and do the assigned problems. Students may not hold up their end of the contract, but they understand the rules. An instructor implementing new instructional approaches with different underlying agreements must make the new expectations explicit. Students will need time to come to terms with the intentional coherence of the methods and messages in the course. This process will require struggle and might elicit rebellion and frustration before students are comfortable with the revised contract. In order to facilitate this process an instructor can explain to students on the first day of class that this will not be a lecture-based class and ask them to read Dana Ernst's blog post danaernst.com/setting-the-stage/. Using new instructional approaches may require new tools to help students succeed in the class such as offering additional office hours, meeting with struggling students early in the term, and coordinating tutoring opportunities.

The next illustration details how an instructor modified a course design to meet his student learning outcomes related to communication.

Classroom vignette: Designing a revision to an established geometry course

Professor Hamilton revised an upper-division modern geometry course that has a proof-based prerequisite, is the final proof-based course required of preservice teachers, serves as an elective for other mathematics majors, and typically enrolls about 15 students. Historically, instructors teaching such a course adopt (a) an abstract approach to neutral geometry that bypasses high school content, (b) an exploratory approach that uses high school content as common knowledge, or (c) an axiomatic approach that reconsiders the common background. The last time he taught the course, Professor Hamilton identified a number of challenges:

- Some students required more writing experiences in order to build their written communication skills.
- Some students were more successful with work completed outside of class.
- Some students requested a reference text.
- Finally, despite his emphasis on epistemological issues related to formal, axiomatic systems in mathematics, some students were overwhelmed by the discussions of these themes.

For the redesigned course, Professor Hamilton adopted an axiomatic approach using David Clark's, *Euclidean Geometry: A Guided Inquiry Approach* (Clark, 2012). Throughout the redesign process, he kept in mind that most of the content of a geometry course is familiar to undergraduates and that this course is primarily intended for preservice high school teachers. He also remained cognizant of the position of the course in the overall program curriculum. In order to emphasize oral communication as well as skill development, the classroom design incorporated inquiry-based learning methods. The course centered on a semester-long activity intended to address the challenges listed above while simultaneously elevating the related learning outcomes. This activity required students to collaborate in writing a geometry textbook. The student-written text consisted of students' proofs, responses to questions from Clark's text, and discussions related to philosophical issues related to formal, axiomatic systems. The course incorporated a wiki to support asynchronous student collaborations outside of class.

Important aspects of the activity design included

1. Each student published at least one polished proof to the wiki after each class meeting. The polished proofs were graded for mathematical accuracy as well as for the quality of communication, and this encouraged students to process class discussions about proof presentations and translate them into written form. Publishing their proofs allowed students who did not always shine during in-class discussions to shine outside class via their written work.
2. The student-authored text served as a reference textbook for students to use during and after the course. This aspect of the course helped students learn to read reference texts more critically and gain an appreciation for the choices textbook authors must make when organizing content.
3. The wiki served as a model for an axiomatic system with hyperlinks acting as logical dependence. Moreover, the habit of only quoting published proofs helped students distinguish the notions of "true" and "proven" as an expert would.

In the course redesign the professor reflected on the overall design of the course and considered the interactions among the course components rather than only considering each component in isolation. He also reflected on students' perceptions about the course design choices and how he might help students understand the choices he had made. For example, students might initially be overwhelmed by the idea of creating a textbook in this course. The task might seem tedious, pointless, and distracting to students. They might perceive it as an additional, meaningless component of an already demanding course. Thus, the instructor must be transparent about the rationale for various course components and proactively address such anticipated issues.

In later iterations of the course, Professor Hamilton added the feature of concept maps (i.e., networks used to help represent and organize subject knowledge) built by each student. These maps illuminate aspects of the course that have been unclear to students and inform revisions in future iterations of the course.

DP.3.3. Embracing opportunities

Although design practices can present challenges, they also present numerous opportunities for improving outcomes of a course, including the following areas

- **Collaboration**. Working relationships can be strengthened and expanded to include additional colleagues.
- **Engagement**. New instructional approaches can increase student-to-student and student-to-instructor interactions.
- **Flexibility**. New instructional approaches and the associated tools and skills instructors develop can be employed in other courses.

- **Differentiated instruction**. New instructional approaches can accommodate students with diverse learning styles.

Collaboration

One of the greatest opportunities associated with engaging in design practices is the possibility of professional collaborations with colleagues both inside and outside the institution. Whether it involves an entire department redesigning a course or just a single instructor considering a new teaching approach, the design process offers an opportunity to build new working relationships. Furthermore, instructors can share their experiences with colleagues at conferences or workshops and can learn from others as well. Such venues provide an opportunity to meet colleagues with similar interests who might become future collaborators.

Engagement

One of the main reasons mathematics instructors try new teaching approaches is to engage and motivate students. However, instructors may find that teaching strategies that engage students also reinvigorate their own interest in a course. After teaching the same course for years, redesigning a course can revitalize an instructor's teaching, especially if they employ a new delivery method.

Flexibility

Another benefit of engaging in design practices is the flexibility to adopt and adapt the practices in other courses. Though it is often prudent to implement new approaches on a small scale in one course at a time, many of the tools an instructor develops will be effective in other courses. In general, design practices are broadly applicable, the processes require less time and effort with increased experience, and many instructors find themselves gradually using the new practices in all of their classes.

Differentiated instruction

Differentiated instruction is a teaching approach that requires instructors to intentionally plan for student differences to facilitate learning for all students. In a differentiated classroom, instructors divide their time, resources, and efforts to reach more effectively students with various backgrounds, readiness and skill levels, and interests. According to Tomlinson (1999), principles for differentiated classrooms include attending to student differences, respecting students by honoring their differences and commonalities, not treating all students the same, ensuring assessment is ongoing and diagnostic, and modifying content, processes, and products as the instructor proceeds through the class.

The design practices discussed throughout this guide lend themselves to differentiated instruction. If formative assessment provides an instructor with a true sense of the differences and commonalities among the students in the class, instructors can differentiate activities in ways that allow each student to reach their potential. Although instructors tend to use differentiating instruction to focus on helping struggling students, they should also be mindful of students who are excelling in the class. These students also deserve attention, and appropriate challenges can serve to heighten their interest in the content.

DP.4. Theories of instructional design

Design practices are generally informed by theories relevant to the field of mathematics education. We present a brief introduction to some of the theories in this section. We do not assume instructors will employ them exactly as described. We simply provide some background on the tenets of the theories that may help inform the process for interested parties. In the brief summaries that follow, we omit some nuances and details of the development of the theories, but we provide references for readers interested in learning more about each particular theory.

DP.4.1. Backward design

The backward design process requires that instructors first articulate both short- and long-term goals for their students. Instructors then identify evidence that would be useful in determining whether or not students have achieved each goal and create activities that will produce such evidence. Instructors must ensure the goals, activities, and assessments are in alignment. As such, formative and summative assessments should be considered in conjunction with course design rather than as a separate component.

The backward design process is based on Fink's taxonomy of significant learning, which consists of six dimensions: foundational knowledge, application, integration, human dimension, caring, and learning how to learn. Similar to Bloom's taxonomy, these dimensions do not exist in isolation, and they manifest differently in each classroom or learning environment. This theory embraces the notion that learning extends beyond memorizing facts and procedures, and instructors should design activities to include as many dimensions as possible in order to fully support students learning. A free online resource for this course design process can be found at www.deefinkandassociates.com/GuidetoCourseDesignAug05.pdf.

DP.4.2. Realistic mathematics education

Realistic mathematics education (RME) is an instructional design theory whose central tenets are guided reinvention and a realistic starting point. Guided reinvention encompasses tasks that guide students to reinvent mathematical concepts and procedures with assistance from an instructor or someone else more knowledgeable than the students. The belief is that if students reinvent the mathematics, then they own the mathematics and are less likely to believe the bearer of knowledge is solely the instructor. Such a philosophy supports both content and affective goals for student learning.

The second tenet of RME is that the mathematics must be grounded in a realistic starting point that is experientially relevant to the students. In other words, even if the setting is not real, it should be meaningful to the students. For example, in an RME based differential equations course, students reason about solutions to differential equations using the idea of an "ideal" fish population that grows continuously. Even though the goal in many mathematics courses is to teach abstraction, tying the concepts to something real has the potential to strengthen students' knowledge and ground their abstracted ideas. Realistic starting points should be part of the instructional design practices, and instructors should look for ways to craft starting points that can be carried over to other lessons.

Classroom Vignette. Designing a task for a linear algebra class

As detailed in Wawro, Rasmussen, Zandieh, and Larson (2013), the Inquiry Oriented Linear Algebra materials' development followed an iterative cycle of task design, implementation, and refinement. In the stage "creating the initial task sequence," the designers draw on various sources including students learning outcomes and mathematical ideas that fit into the broader scope of the course. The task designers aim to create tasks that have the potential to facilitate those desired outcomes, drawing on the RME guideline that tasks should have the potential to elicit students' intuitive ways of reasoning about mathematical ideas. These ways of reasoning can in turn be leveraged toward more formal mathematical reasoning. Although the designers cannot be certain how students will engage in the task sequence, they draw upon their knowledge of student reasoning, both as linear algebra instructors and as mathematics education researchers, to design a new task sequence.

Next the designers pilot the task sequence with a subset of students outside the actual class. They video record this set of students attempting the task, and one of the researchers interacts with the students as an instructor might. They review the video data to gain information about how the students engaged with the task. What ways of reasoning were elicited? Which ways of reasoning were productive and which were problematic? Did the task facilitate the development of more formal ways of reasoning about the concept(s) involved? The developers use the information gleaned from analyzing the task's first implementation to inform refinements of the task, and then they use the refined task sequence in a classroom environment, beginning the second iteration of the design research cycle. They continue this cycle of refining the task and implementing the new version until some balanced state is achieved, although the task and its implementation are never completely stable because an inquiry-oriented classroom requires continual responsiveness and adaptation to student thinking.

DP.4.3. Universal design for learning

Universal design for learning (UDL) is a framework based in brain science and designed to improve and optimize learning for all students based on scientific insights into how humans learn (CAST, 2011). There are three primary principles that guide UDL. First, designers should provide multiple means of representation requiring options for perception, for comprehension, and for language, mathematical expressions, and symbols. The second principle calls for designers to provide multiple means of communication, physical action, expression, and executive functions. Executive functions include setting long-term goals, planning effective strategies for reaching those goals, monitoring student progress, and modifying strategies as needed. The third principle calls for designers to provide multiple means of student engagement. This can consist of creating ways to heighten students' interest in the work, providing options for sustaining effort and persistence, and offering options for self-regulation.

There are several other learning theories that are important for design and worthy of further reading. These include action, process, object, and schema (APOS) theory (Dubinsky and McDonald, 2001); models and modeling (Lesh and Doerr, 2003); Piagetian constructivism (Piaget, 1967); and sociocultural theory including the notion of zone of proximal development (Vygotsky, 1978).

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Cross-cutting Themes

The role of technology and the role of equity in teaching and learning mathematics are inherent in each of the three practices detailed in this guide: classroom, assessment, and design. Instead of discussing technology and equity at length within each of the practice chapters above, we have chosen to address them here as themes that cut across all three practices. We encourage instructors to reflect on how these two themes play a role in classroom, assessment, and design practices beyond the examples included in this guide.

Technology and instructional practice

XT.1. Introduction

In today's world, technology is ubiquitous and applicable to many aspects of instructional practice. As such, instructors should continually examine how and where technology fits into their work. Classes may incorporate audience response systems or computer-based explorations, assessments may include online homework and examinations, and course design may depend on specialized classroom technologies or even the absence of a physical classroom. The technological landscape is vast and potentially intimidating but immensely powerful in facilitating student learning.

The central theme of this guide, intentional instructional practice, is especially important in the context of technology. The latest instructional technology fad is not guaranteed to enhance student learning, particularly if it is used in a less than intentional way or in a way that does not align with learning goals, teaching style, or instructor's comfort level with the technology itself. This chapter details ways in which technology may be used for instruction, synthesizes research related to the effectiveness of technology in improving student learning, and includes suggestions for integrating technology into the classroom, design, and assessment practices described in this guide.

XT.2. Uses of technology

The *CUPM Guide* (MAA, 2015) identifies five broad areas in which technology can be used to enhance teaching and learning: exploration, computation, assessment, communication, and motivation. This Instructional Practices Guide provides a lens through which to view these five areas interwoven with classroom, assessment, and design practices in an effort to promote student engagement and learning.

For example, exploring a mathematical object or concept with technology may facilitate student engagement while simultaneously facilitating formative assessment. The computational power of technology can facilitate work on projects set in real-world contexts that in turn may motivate increased student interest and offer more sustained opportunities for collaboration and communication. Such projects can also serve as capstone, i.e., summative, assessments of student knowledge and understanding. Technology can serve as a venue for communication among students and between students and instructors via online chats or phone apps that both engage students and provide formative feedback. Furthermore, online homework or examinations may serve as formative or summative assessments and encourage a higher level of student engagement, particularly with systems that provide immediate feedback to students on their work. Technology can serve as both a learning tool and an assessment instrument depending on how and when it is used. Accordingly, instructors should consider how available technologies support their instructional goals and intentionally design courses to utilize available technology to increase student engagement and improve learning. The variety of technology available can further complicate these goals, and the use of technology will vary greatly from instructor to instructor. Thus, this guide is not intended as prescriptive; rather, it outlines research on the effectiveness of technology in improving student learning and provides information to assist instructors in making informed choices on the use of technology in their classrooms.

XT.3. Effectiveness of technology

The use of modern technology in mathematics instruction might be dated to the advent of the electronic calculator in the early 1970s. The breadth of research on the use of these calculators in the K–12 curriculum is informative for instructors at the postsecondary level as well. The use of calculators is necessarily focused on improving student learning (rather than, for example, assessing student knowledge), and the research on their effectiveness is unambiguous. The National Council of Teachers of Mathematics (NCTM, 2011) endorses their use, based on the finding that "the use of calculators in the teaching and learning of mathematics does not contribute to any negative outcomes for skill development or procedural proficiency, but instead enhances the understanding of mathematical concepts and student orientation toward mathematics" (p. 1). This strong statement is based on several meta-analyses of numerous research studies on the effectiveness of calculator use in improving student learning (e.g., Ellington, 2003, 2006; Hembree and Dessart, 1986).

The strength of this conclusion merits additional discussion. The calculator technology referenced in the statement is, by today's standards, almost trivial, having a very limited set of capabilities. Yet, the existing body of research is significant in breadth, spanning 35 years and hundreds of research articles. The exciting news is that there now exists a definitive canon of literature that pinpoints a central theme underlying effective instructional practice, dating back at least to the electronic calculator:

> **Students must *actively engage* with the concepts they are learning** (CBMS, 2016; Freeman, et al., 2014; Kogan and Laursen, 2014; Laursen et al., 2014).

Thus, when instructors intentionally select mathematical tasks along with appropriate technology to promote student engagement with the material, they have the greatest chance of improving student learning. It is difficult to overstate the importance of this conclusion.

Many other studies have shown enhanced student outcomes based on the use of various technologies such as graphing calculators (Ellington, 2006), audience response technology (i.e., "clickers"; see e.g., Cline and Zullo, 2012; Simelana and Skhosana, 2012; Stowell and Nelson, 2007), and online homework and examinations (e.g., Hirsh and Weibel, 2003; LaRose and Megginson, 2003). If instructors use technology in ways consistent with the central themes of student engagement and intentional task selection, they can expect to see improved student learning outcomes. We argue that questions such as "Does this technology fit into this learning environment?" and "How should I use this technology in that class?" are the wrong questions with which to begin. Instead, instructors should begin by considering their learning goals and their own comfort level with various technologies, then ask "Which technologies can help me accomplish my goals and create an engaging learning environment?" Fundamentally, this question motivates the remainder of this discussion.

XT.4. Technology incorporated into instructional practice

Given the breadth of the applications of technology that are available and the speed with which they change, this guide does not attempt to provide an exhaustive list of current applications. Rather, it offers illustrative examples of the use of various technologies that are easily adaptable to different environments and other technologies with comparable capabilities.

XT.4.1. Technology and exploratory activities

Many technology applications appropriate tools for engaging students in exploring mathematical concepts. This type of technology can experientially provide students a view of mathematical structures and relationships, similar to the way they observe the manifestation of physical laws in a lab science course. In other cases, students may use the the technology to consider problems that are intrinsically interesting that in turn may increase student motivation and engagement with the material.

Example 1. The first vignette in section DP.2 provides an example of using technology to promote student exploration of mathematical concepts. Many courses include a student learning objective to demonstrate understanding of the relationship between a mathematical expression (e.g., a formula or equation) and the quantities appearing in the expression. Technology is well-suited for actively exploring this type of relationship. The vignette describes how students use a spreadsheet application to explore ways in which different terms in a formula affect the output.

Example 2. A similar example from a differential equations course is shown below. A learning objective for the course is that students will demonstrate understanding of the relationship between solutions to a nonlinear system and to its linearization at a critical point. Because solving a nonlinear system is, in general, not analytically possible, this is an opportunity for students to utilize technology such as Maple, Mathematica, Matlab, or Sage to investigate the relationship. The following set of exercises can be completed with such technologies.

Differential Equations lab

A. Model

A van der Pol oscillator is a model of an active RLC circuit with a nonlinear resistor that dissipates energy when the amplitude of the current is high, and pumps energy into the system whenever the amplitude of the current is too low. An equation modeling such a circuit is

(a) $$x'' + \mu(x^2 - 1)x' + x = 0,$$

where μ is a positive constant.

B. Prelab activities

Before coming to lab, complete the following activities:

1. Write the van der Pol oscillator (a) above as a first-order system in x and $y = x'$.
2. Show that the only critical point (i.e., values of x and y for which both x' and y' are zero) for the system from activity 1 is the point $(0, 0)$.
3. Near $(0, 0)$, we can linearize the system by assuming x and y are very small. If this is the case, then terms like x^2 and $x^2 y$ are very, very small—so small that it is reasonable to drop them to obtain a linear system. Find a linear system approximating your system from activity 1, and write it in matrix form.

C. Lab activities

Ignoring, for the moment, the fact that part A above indicates μ must be positive, consider the case $m = -1$.

1. Find numerical solutions to the nonlinear and linear systems you found in the prelab activities for several initial conditions close to the origin. For each initial condition, plot a component plot of x as a function of time with the solutions to the linear and nonlinear system on the same graph.
2. Find numerical solutions to the nonlinear and linear systems for larger initial conditions. (Note that for this value of μ, you must carefully choose the range of t values for the nonlinear system when the initial conditions are large enough.)

3. Finally, graph two phase portraits, with x on the horizontal axis and y on the vertical. In the first, include all the solutions to the linear system you found in activities 4 and 5, and in the second, include all the solutions to the nonlinear system.

Repeat activities 4, 5, and 6 for $\mu = 1$. Note how your results are similar and different.

D. Lab writeup

Write a 1–2 page summary that explains how the solution to the linearized system you obtained in pre-lab activity 3 is similar to and different from the solution to the nonlinear system from pre-lab activity 1 and where the linearization allows us to say something about the solution to the nonlinear system. Include graphs of solutions to the systems for different initial conditions and an explanation of why the results make sense given how you obtained the linear approximation to the nonlinear system.

For students to productively engage in tasks using technology, the learning environment must promote the value of this type of student engagement, and instructors must clearly articulate their expectations of students. In addition, instructors must structure tasks in ways that promote productive collaboration among students. Finally, the task design must reflect the learning goals the task is intended to accomplish.

XT.4.2. Technology and formative assessment

There are many ways in which technology can be used for formative assessment. Following are two examples that build on vignettes from the Assessment Practices chapter.

Example 3. In Vignette 1 in section AP.2.1, Dr. Doe gives an in-class quiz over content from the previous class period and uses the results to adjust what she covers in the subsequent class period to adequately address students' difficulties with trigonometric identities. The following week, she decides to implement a similar feedback loop using her online homework system (e.g., ALEKS, MAA WeBWorK, MyMathLab/MyStatLab, WebAssign) rather than an in-class quiz to provide feedback on students' understanding prior to class. (She might instead have used her course management system, e.g., Blackboard, Canvas, Moodle, Sakai.) Dr. Doe creates a writing assignment, due a few hours before class, that requires students to answer simple questions over material they were assigned to read prior to class. Reviewing students' responses before class will provide her a good sense of the concepts she need not review and those she should cover more in-depth.

Example 4. A similar application of technology is illustrated in the vignette in section AP.6.2. Professor Ordinal utilizes a classroom polling system to assess student learning in real time. This both increases student engagement in class and provides formative feedback to Professor Ordinal on what students understand and what their misconceptions are. In this respect, the polling system serves the same role as Dr. Doe's quiz and writing assignment (Example 3 above). Instructors who understand their students' current knowledge states are better positioned to facilitate increased student learning.

Example 5. Instructors can use formative assessment to encourage student ownership of their own development. A common example is a skills or "gateway" test in a first-semester calculus course with a learning goal of students mastering various fundamental techniques of differentiation. Following is an example of a technology-based test:

Calculus differentiation gateway test

1. Set-up

The differentiation gateway test is administered through an online homework system and exists in two identical versions. Both versions allow students as many attempts as they wish, consist of problems drawn from the same problem bank, have a fixed time limit, and allow a single submission. One version, called the "Practice Gateway Test," allows student access from any location and requires no proctor authorization. The other version, the "Proctored Gateway Test," allows student access only in a specific computer lab where proctors are available to verify the students' identities and provide a required password.

2. Syllabus description

This is a first course in calculus, and its primary goal is for you to obtain a rich, conceptual understanding of the fundamental ideas of calculus. Much of the work you will do in this course focuses on conceptual ideas and will require you to reason about concepts on many levels. However, there are some basic skills involving differentiating functions that you will also learn in this course, skills that every student taking first-semester calculus can and should master. Toward this end, you are required to take a differentiation gateway test on which you must correctly complete at least six of the seven problems within 30 minutes in a computer lab with a proctor present. You have two weeks to complete this task. You may take the test in the lab up to twice per day, but you must review your first test with one of the tutors in the Math Learning Center before returning to the testing lab to take the test the second time that day. You may practice the test as many times as you like, from wherever you like, by logging into the online homework system for this course and clicking the "Practice Gateway Test" assignment. I recommend you complete the practice test as many times as necessary until you can reliably pass it before you attempt it in the proctored computer lab. Failure to complete the gateway test successfully by the deadline will lower your grade in this course a full letter grade.

3. Technical details

The test consists of seven questions, each of which is drawn from a test bank associated with a specific type of differentiation problem. One requires using the product rule, another requires implementing the quotient rule, another requires differentiating an expression with symbolic parameters, etc. If a student takes the test twice in the lab and does not pass it, they are required to take a break of at least one full day, increasing the likelihood that they will use the practice test to hone skills they have not yet mastered.

In this case, the technology serves two purposes: it minimizes the human effort required to manage the logistics of the testing process, and it provides students immediate formative feedback when they complete the test as well as correct solutions demonstrating how each problem could have been solved. It also frees up instructor time from grading, allowing more time for engagement with students outside of class (e.g., during office hours).

XT.4.3. Technology as a tool

There are fundamental tasks instructors must complete in every course they teach. For example, distributing the syllabus and other resources, assigning homework, communicating with students outside of class, etc., all of which can be facilitated by technology. Instructors may distribute hard copies of materials to students or post them electronically using a course management system (CMS) such as Blackboard, Canvas, Moodle, or Sakai. Instructors may assign written homework, i.e., "pencil-and-paper" assignments, or may make assignments to be completed online via systems such as ALEKS, MAA WeBWorK, MyMathLab/MyStatLab, and WebAssign. They communicate with students in class and during office hours and may also use an online chat, forum, or discussion board application as a stand-alone tool or perhaps embedded in a CMS.

In all cases, instructors should choose technology tools based on functional capabilities of the tool and how the tool impacts student learning. Students are often inclined to first check their CMS for course information and resources so distributing materials via a CMS can be a highly effective way to provide access and encourage student use of the materials. A chat application provides the means for an instructor to engage in real-time discussions with students at times when the instructor is not available on campus. A forum or discussion board application can promote communication between the instructor and students as well as among the students themselves, creating a classroom community in which all participants are working to advance learning for all.

Example 6. Section 6.1 of the Assessment Practices chapter includes a discussion of the advantages and disadvantages of online homework systems in some detail. Below is an example of how an instructor might include online homework in a course.

Math 314 course components

Online homework: Homework administered through our online homework system will cover most of the course content and will be due most Wednesdays as indicated in your day-by-day syllabus schedule. On each problem of each homework assignment, you are allowed up to six attempts, and the system provides immediate feedback on the correctness of each attempt. Once the assignment has closed, you can see the correct answers to the problems along with detailed solutions.

Reading homework: Brief written responses to reading questions will be due approximately daily. These short assignments comprise three questions over the material you are to read before class. The questions are chosen to highlight important formulas and ideas and are due slightly in advance of each class period so I can use them to determine which topics need more or less coverage in class.

Written homework: Approximately weekly, on Fridays, a written homework assignment will be due. These require solutions written out with full explanations. The problems are intentionally chosen to be more involved and conceptual than the online homework problems and are designed to give you the opportunity to explore the course content more deeply. Your lowest written homework score will be dropped from your course grade calculation.

This example illustrates how the online homework system is used to complement other assessment methods and promote student engagement with course content. The strength of the technology—providing immediate feedback to students and aggregated results to the instructor—is exploited and frees up "grading time" for shorter, traditional assignments that focus on deep, meaningful exploration of concepts as well.

We acknowledge the challenges to academic integrity that may be exacerbated by the use of technology for instruction. For example, a student may use their smartphone to send a copy of the exam to a friend who can provide solutions to them during the exam. In a multi-section course taught by the same instructor, a student may use their phone to share a copy of the exam with a student in the subsequent section to allow that student time to prepare answers prior to the exam. The internet provides free, instantaneous access to tools such as Wolfram|Alpha as well as the ability to communicate with anyone else anywhere in the world who has internet access. There are discussion boards where students can obtain solutions by simply posting even the hardest proof that might be assigned in an advanced graduate mathematics course. There are online tutors who provide solutions to any homework assignment an instructor might make. Obtaining answers to many problems from undergraduate mathematics courses require nothing more than a simple internet search for the text of the problem. The reality is that the ubiquity and power of technology means any assignment completed in an unproctored environment can be completed without a student doing any of the expected work. A positive consequence of this new reality is that student access to answers to skill-based, procedural tasks opens the door for more sophisticated and diverse assessments that promote deeper conceptual engagement and understanding. And that, after all, should be our primary goals: basic skill development and procedural fluency along with deep understanding of fundamental mathematical concepts.

XT.5. Practical implications

This chapter echoes the themes of intentionality and appropriate task selection that run throughout this guide. In utilizing technology in learning environments, instructors must be intentional in their design to ensure the technology helps create classroom environments conducive to student learning and serves as a powerful formative and summative assessment tool. The Classroom Practices chapter includes a detailed discussion on selecting appropriate tasks to promote student learning, and the Design Practices chapter poses appropriate questions to guide the design of learning environments, all of which directly apply to selection and implementation of technology in the classroom.

Instructors must also consider issues of equity and inclusion in the context of intentionally using technology for instruction. For example, using technology to show three-dimensional graphs in a multivariate calculus course raises the question of accessibility for students with vision impairments. A possible solution would be to use 3-D printer to generate an object the student can hold in their hands while it is discussed in class. Using an online homework system or discussion board application might cause accessibility issues for students with limited or no internet access at home. Ensuring students have access to the internet on campus will likely solve the problem for some students but might not be sufficient for students who have family or work commitments that prevent them from spending additional time on campus outside of class. Instructors must be intentional in their use of technology and work to create inclusive, non-threatening environments for all their students. Further discussion of equity and inclusion issues is included in the in the equity section of this chapter and in the Classroom Practices chapter of this guide.

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Equity in Practice

XE.1. Introduction

The number of mathematics degrees awarded at the undergraduate and graduate levels provides insight into the impact of institutional cultures and instructional practices on women and historically underrepresented groups in science, technology, engineering, and mathematics (STEM). In 2012, only 20% of bachelors, 18% of masters, and 8% of doctoral degrees in mathematics were awarded to black, Latinx, native American, native Alaskan, and Hawaiian students combined (National Science Board, 2014) despite the fact that these racial groups composed approximately 30% of the U.S. population at that time. Further, the 2010 survey of mathematics departments conducted every five years by the Conference Board of the Mathematical Sciences (CBMS) indicated members of these underrepresented groups composed only 9% of the full-time mathematics instructors (CBMS, 2013); while women made up 29% of these full-time instructors, only 3% were women of color.

Research has revealed additional and sometimes hidden stressors placed on women and students of color as they navigate undergraduate and graduate mathematics (Herzig, 2004; McGee and Martin, 2011a; 2011b). McGee and Martin (2011b) detailed how academically successful black undergraduates pursuing mathematics and engineering majors faced racial stereotypes of low ability and underachievement. Experiences in undergraduate mathematics classes have also been shown to contribute to women's decisions to leave STEM fields despite the fact that they are well-prepared and fully capable of succeeding in these fields (Ellis, Fosdick, and Rasmussen, 2015; Kogan and Laursen, 2013). Such research suggests our community needs to critically examine factors well beyond students' academic preparation and achievements in our quest to increase students' success in STEM. Such factors include implicit messages our course design and teaching practices send to students regarding what mathematics is and who "belongs" in mathematics. Adiredja and Andrews-Larsen (2017) provides a more detailed review of research in postsecondary mathematics education related to equity issues at the institutional level.

Fixation in higher education on low achievement rates among women and students of color in mathematics, coupled with erroneous notions that mathematical ability is innate and fixed, contribute to the prevalent deficit perspective of these underrepresented groups, especially among a predominantly white teaching force (Battey and Leyva, 2016; Harper, 2010; Valencia, 2010). Such deficit perspectives, that focus on what students cannot do, often result in instructors reducing the rigor of mathematical tasks and assessments, avoiding instructional strategies that engage students in higher-level reasoning, and failing to build positive relationships with students from these groups (Battey, Neal, Leyva, and Adams-Wiggins, 2016; Ladson-Billings, 1997; Lubienski, 2002). It is incumbent upon us to consider classroom, assessment, and design practices that affirm our students and provide equitable access to rich mathematical learning opportunities for all. We must challenge the deficit perspective among the broader mathematical sciences community and help our colleagues broaden their notions of mathematical competence and success while still maintaining high levels of rigor and standards of performance.

XE.2. Definitions

XE.2.1. Four Dimensions of Equity

Gutiérrez (2009) offers a framework to define and conceptualize equity in mathematics education. Her model involves four key factors: access, achievement, identity, and power (see Figure 1). Access and achievement occupy the "dominant axis" as these dimensions of equity focus on supporting students to participate in the existing dominant culture and practice of mathematics. Addressing issues of access and achievement support students in learning the rules of mathematics and successfully "play(ing) the game" (Gutiérrez, 2009, p. 6). Attending to access means ensuring all students have access to physical and intellectual resources to learn mathematics (e.g., good instructors, rigorous curricula, opportunities to think critically about mathematics). Achievement focuses on student learning outcomes as traditionally measured (e.g., scores on exams, persistence in mathematics, majoring in STEM).

Dimensions of Equity

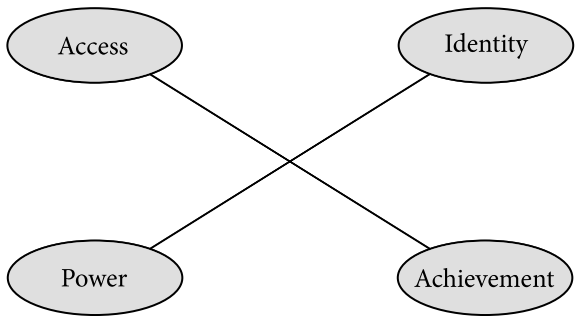


Figure 1. Diagram adapted from Gutiérrez (2009).

Identity and power occupy the "critical axis" as these dimensions of equity focus on supporting students to become critical participants who have the potential to "change the game" of mathematics (Gutiérrez, 2009, p. 6). These two are the most transformative of the four dimensions in terms of their potential to affect monumental change in mathematics education. Attending to identity means recognizing ways in which the constellation of social identities students bring (e.g., race, gender, social class) can be a resource in learning. We must educate ourselves and remain ever cognizant of the ways students' social identities impact their participation in the classroom. We must acknowledge ways in which these identities serve to include or exclude students based on the prevailing view of various identities in the context of learning mathematics. For example, the stereotypical view that all Asians are good at mathematics affirms that Asians "belong" in mathematics but excludes other racial identities (Martin, 2009) and can lead to exclusion of students from groups that have been historically marginalized (e.g., black students, see Nasir and Shah, 2011). Further, this stereotype can lead to the erasure of the needs of particular Asian groups that have had limited access to educational opportunities (e.g., 38% of Hmong-Americans have less than a high school degree compared to the 13.4% national average, Center for American Progress, 2015). Stereotypical hierarchies of intelligence are damaging for *all* students.

Attending to issues of power means examining the degree to which learning disrupts or challenges the existing distribution of resources and influence in the classroom as well as in society. This distribution is often unequal in terms of race, gender, and social class (Gutiérrez, 2009). Thus, attending to power means asking questions such as, "Who benefits from the teaching of mathematics and to what end?" or "Is this mathematics empowering students or does it maintain the status quo?"

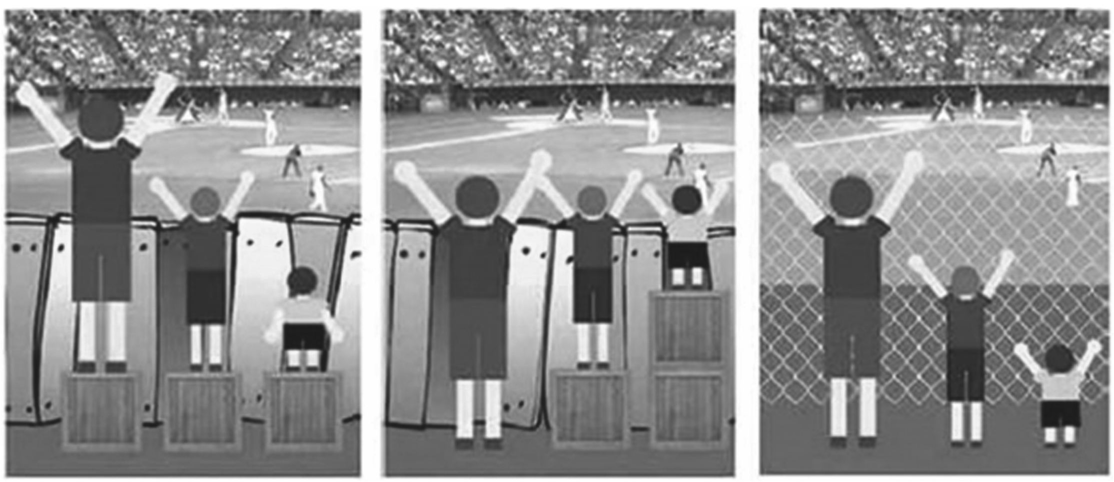
Challenging existing power dynamics can be achieved by exploring the use of mathematics to critique social and political issues (Gutstein, 2003). For example, Tufts University hosted a workshop for mathematicians on the "Geometry of Redistricting" to analyze the legality of gerrymandering.

Gutiérrez notes that the two axes of equity are often in tension with each other. For example, supporting students to successfully participate in the current practice of mathematics might inadvertently ignore aspects that exclude some students from participating. Exploring a non-traditional use of mathematics or challenging an existing power distribution might lead to exclusion of some students in the current culture of mathematics. Gutiérrez's framework can help guide us in thinking critically about ways to broaden access to mathematics and in designing inclusive and equitable mathematics classrooms where all students can thrive.

XE.2.2. Equity, Inclusion, and Systemic Barriers

A primary tension that comes into play in the process of addressing issues of equity in undergraduate mathematics education is distinguishing *equity* from *equality*: *equity* focuses on social justice whereas *equality* focuses on sameness (Gutiérrez, 2002). Sameness refers to a response in which all students are treated the same regardless of their backgrounds and skills. This type of context-free[1] approach offers the illusion of fairness but ignores the critical roles that students' experiences and identities play in their education. Gutiérrez asserts, "To redress past injustices and account for different home resources, student identities, social biases, and other contextual factors, students, in fact, need different (not same) resources and treatment to reach fairness" (2002, p. 152). Context-free approaches ignore these factors and continue to privilege students from the dominant groups.

Equality versus Equity



In the first image, it is assumed that everyone will benefit from the same supports. They are being treated equally.

In the second image, individuals are given different supports to make it possible for them to have equal access to the game. They are being treated equitably.

In the third image, all three can see the game without any supports or accommodations because the cause of the inequity was addressed. The systemic barrier has been removed.

Figure 2. Equity v. Equality (image source: culturalorganizing.org/the-problem-with-that-equity-vs-equality-graphic/).

The first two images of Figure 2 highlight the critical need to attend to students' different contextual factors (here, their heights). The third image illustrates the removal of the barrier (the wooden fence), thereby removing the need for accommodations, which results in equity.

Removing barriers is the real key to equity and inclusion. Achieving equity in undergraduate mathematics education is a formidable task that will require philosophical shifts in the way our community views the accessibility of mathematics, particularly as a social justice issue. We must first identify the systemic barriers inherent in higher education in general, and in mathematics education specifically, and then devise strate-

1 We use the term "context-free" instead of "color-blind" or "gender-blind" to describe the lack of attention to individual's backgrounds. The term "color-blindness" has been useful in describing beliefs about freedom from racial bias and led to powerful critiques about such beliefs in a racialized society (Bonilla-Silva, 2003). However, the terms discriminate against people with visual disabilities by erasing or delegitimizing their existence and experiences (Colorblind, 2011).

gies for removing these barriers for our students. All our students deserve access to mathematics.

We must utilize effective methods for supporting students in becoming better learners as we work to change departmental and institutional processes, policies, and cultures that act as barriers to student success. We must ensure all students have the opportunity to experience the rigor, practicality, elegance, and beauty of mathematics (dominant axis). Perhaps more critically, we must examine mathematics as an institution with its own set of norms, values, and practices and identify ways to provide a more inclusive, affirming environments for students, particularly students from underrepresented groups (critical axis). For example, how can we problematize acceptable expression of mathematical ideas when students are still learning the formal mathematical language? How do we conceptualize rigor in different stages of learning for our students? Certainly, there is no implication here that lowering our expectations and level of rigor is in any way acceptable. Rather, the onus is on our community to maintain high academic standards as we consider systemic barriers in learning mathematics.

The "growth versus fixed mindset" theory of intelligence (Dweck, 2006) can serve as an instructive example in this context. This theory is appealing to the education community because of its explanatory power, but it has limits of which instructors must remain mindful. In utilizing the theory to effect positive change toward increased student learning and success, instructors must remain cognizant of the potential to inadvertently limit access to mathematics for students. How might that occur?

The theory posits that individuals with a "growth mindset" are more likely to persist and succeed in the face of challenging tasks compared to individuals with "fixed mindset." That is, those who view intelligence and ability not as inherent qualities but rather as malleable qualities are more likely to improve their skills and understandings over time. Those with a fixed mindset are less likely to persist on a challenging task. A recent publication by Boaler (2015) details specific applications of the growth/fixed mindset model in mathematics, such as the role of struggle in expanding students' knowledge and abilities. Helpful questions arise for reflection as we consider mindsets in mathematics students: What messages do we send students about the field of mathematics? To what extent do we view mathematical ability as innate in students? To what extent do successful learners of mathematics experience struggle and need time to make sense of mathematics?

The danger can arise when we inadvertently treat students' adoption of growth mindset as the only means to address inequities. It is counterproductive when an instructor views students as "change-worthy" and focuses on changing the students while ignoring the systemic barriers that perhaps prompted the fixed mindset view students have of themselves. Solely focusing on students' mindset ignores the impact of systemic oppression (e.g., racism) on students' lives and educational experiences. McGee and Stoval (2015) have extensively discussed a similar fixation around the notion of "grit" in education and its failure in accounting for the impact of racism on the mental health of black students. The growth mindset model is a useful concept but should not be viewed as a singular quick fix to the very complex issue of equity and inclusion in mathematics. Such an approach needs to be coupled with continued work to remove systemic and institutional barriers for all students to be successful. We now offer some principles that can assist with the implementation of the specific suggestions from earlier chapters and begin the process of addressing equity in the classroom.

XE.3. Higher-order equity-oriented principles

XE.3.1. Social discourses and narratives impact teaching and learning

Established social discourses and narratives around social identities (e.g., race, gender) and intelligence impact students' sense of belonging and their opportunities to participate in the classroom (Leyva, 2016; Nasir and Shah, 2011). Deficit narratives about students, particularly black and Latinx students as academically and intellectually inferior, limit access to educational opportunities (e.g., who is called on in class, who is advised into STEM majors). These narratives can also place unnecessary cognitive burdens on students in learning environments, particularly for students operating under "stereotype threat." Steele and Aronson (1995) identified and defined stereotype threat as a situational predicament in which individuals are at risk of confirming negative stereotypes about their group because they will be judged based on negative stereotypes about their group rather than their own merits. These researchers investigated the effects of stereotype threat on students when performance was linked to intelligence.

The researchers found that black college freshmen and sophomores performed worse on verbal tests in an academic environment than white students when their race was emphasized. The typical race gap in achievement emerged when stereotype threat was activated via a reminder of a negative stereotype about their group's intelligence. White students performed at the same level under both conditions, but black students performed as well or better than their white peers in the absence of stereotype threat. They found similar patterns in test performance between women and men. Follow-up studies suggest that in situations where their ability is being evaluated, stereotyped students carry an extra weight on their minds related to the stereotypes about their group.

XE.3.2. All students are capable of learning mathematics

There is no special "mathematics gene," only social valuation of skills that align better with the traditional methods of instruction in mathematics (e.g., passive lecturing). Ease in understanding mathematics is not an inherent personal quality but a product of prior opportunities and social positioning. Similarly, students' and instructors' behaviors and dispositions are in part a product of socializations. Their knowledge is influenced by their environment and distribution of resources. Categorizations of students as "mathematics students" versus "non-mathematics students" or "slow" versus "fast" are artificial, limiting, and not conducive to learning. Research has shown that the way teachers label and talk about students impact how they respond to students' difficulties in the classroom (Horn, 2008).

Instructors must deliberately adopt an anti-deficit perspective on students and their knowledge in order to recognize that all students have the ability to contribute in the classroom. Misconceptions and errors in student thinking are a natural part of learning. The fixation on remediation is deficit-oriented, undermines student progress, and hinders the development of mathematical identity. The value of students' ideas should not be solely based on proximity to the norm.

XE.3.3. The importance of fostering a sense of classroom community

A critical aspect of learning mathematics is participating in mathematical discourse in an environment that supports students sharing and critiquing their own and each other's work. The work of teaching is not an activity solely between a student and a teacher. Student participation in the classroom is influenced by the distribution of authority, status, and power among all participants in the classroom. Authority, status, and power are all influenced by students' social identities. Experiencing other students as resources in learning fosters students' connections to the classroom community. This requires a safe environment for students to share partial understanding, communicate freely with other students, and build on each other's knowledge.

XE.4. Attending to equity

XE.4.1. An illustration: Students with disabilities

Most mathematics instructors have had experience attending to equity issues in the classroom related to providing accommodations for students with disabilities. Instructors recognize the importance of providing accommodations to facilitate the learning process and ensure students with disabilities are not further marginalized in their learning experiences. These students have to navigate learning environments differently from other students. We recognize that we are not experts on the particular needs of a student. For example, we cannot treat all students in a wheelchair the same way because they will likely have different needs. We rely on the assistance from both the student and the office for disability services on campus to understand the student's particular needs. We as instructors work in collaboration with students to create the most supportive and inclusive learning environment. We understand that an inclusive classroom environment would benefit all students in the class. For example, speaking more slowly in the classroom would help accommodate an interpreter for a student with hearing impairment as well as provide other students more time to process information. For additional information on students with disabilities, see section 2.7 in the Design Practices chapter.

Some of these ideas are helpful as we consider an equity-oriented approach to teaching for other marginalized students. We support students by focusing on the needs of individual students and recognizing their histories and positioning in society. We do not treat all marginalized students the same way. The students are an important resource in learning about their needs. We need to work in collaboration with them in providing the most supportive learning environment. We can also draw on resources outside of our own departments (e.g., Office for Diversity and Inclusion) to best serve students. For example, these offices in Student Services are typically equipped to assist in issues related to microaggressions—e.g., everyday communicative actions or verbal expressions that may or may not intentionally slight target or marginalized individuals such as students of color (Sue, 2010)—or other challenging conversations in the classroom. Ultimately, our students are the best resource in our effort to create a more inclusive classroom environment that serves all of our students.

XE.4.2. Critical need to attend to developmental mathematics

As we consider instructional practices in the context of different topics in mathematical and types of institutions, one particular issue that requires careful consideration is developmental mathematics. The national pass rates in developmental mathematics courses in both two- and four-year institutions are disconcertingly low. This has prompted scholars to investigate factors associated with student success in such courses (e.g., Fong, Melguizo, and Prather, 2015) and to recognize the value of curricula focused on quantitative reasoning and statistics more than algebra (Hoang et al., 2017). Furthermore, poor performance in developmental mathematics courses is correlated with dropout rates and low transfer rates (Bonsangue, 1999; Fong et al., 2015). Multiple failed attempts by students to pass these courses place undue financial burdens on both students and states (Fong et al., 2015). Even more disconcerting, non-traditional students and various underserved populations are overrepresented in these courses. For example, Larnell (2016) cited studies that confirm the disproportionate number of black students in these courses (e.g., Attewell, Lavin, Domina, and Levey, 2006; Bahr, 2008).

The principles and practices outlined in this guide are particularly relevant in the context of developmental mathematics courses. Many of the students in these courses are there precisely because our traditional teaching practices (e.g., passive lectures) have failed these students. Yet the CBMS 2010 survey reports that these courses are dominated by traditional lectures. While a minimal amount of traditional lecturing can have a place in an active-engagement environment, the evidence-based practices detailed in this guide offer benefits and support for students in developmental mathematics courses. A documented barrier to instructors adopting innovative, evidence-based teaching practices is the perception that students in lower level courses are unable to engage in deep mathematical reasoning, which brings us back to the notion of anti-deficit perspectives on students and their knowledge.

XE.4.3. Conclusion: Anti-deficit perspective and focus on excellence

Research has consistently shown the positive correlation between instructors' high expectations of students and student success in mathematics (e.g., Asera, 2001; Delpit, 2012; Gutiérrez and Dixon-Román, 2011; National Collaborative on Diversity in the Teaching Force, 2004). Course design as well as instructional and assessment practices framed by high expectations and anti-deficit perspectives have a positive effect on how students see themselves in relation to mathematics. In her study of instructors supportive of black students, Ladson-Billings (1995) found that the common factor across all instructors was their anti-deficit perspective.

Some of the documented curricula, programs, and pedagogical approaches shown by research to successfully support underrepresented populations in mathematics are strongly driven by anti-deficit perspectives about students. For example, the Treisman Math Workshop program, which originated at the University of California, Berkeley, dismissed the narrative that black and Latinx students lack resources and motivation to do well in mathematics (Treisman, 1992). The workshop was designed as an honors program to provide students with rich learning opportunities to engage critically with mathematics (Asera, 2001). The Meyerhoff Scholars Program at University of Maryland Baltimore County is one of the few programs that focuses on underrepresented students' success in STEM (Miller, Ozturk, and Chavez, 2005). Similarly, inquiry based learning (IBL) efforts have been shown to "level the playing field" between male and female students by building on the premise that all students are capable of engaging in higher level mathematical practices such as conjecturing and generalizing (Laursen, Hassi, Kogan, and Weston, 2014).

These innovative programs and practices also impacted the development of students' mathematical identities and redistributed power in students' experiences with mathematics. For example, Oppland-Cordell and Martin (2014) found that in an Emerging Scholars Program calculus workshop, students sharing their mathematical work publicly recalibrated peers' perceptions of intelligence related to race, gender, and other social identities. Through observing strong mathematical work by fellow students of color, Latinx students recognized their own excellence in mathematics and challenged existing narratives about the perceived superiority of their white and Asian peers in mathematics. Hassi and Laursen (2015) documented how the implementation of IBL instruction in calculus courses resulted in empowerment of female students who then perceived themselves as mathematically competent and expressed interest in future IBL mathematics courses at higher rates than female peers in non-IBL courses. Anti-deficit perspectives shape socially-affirming forms of course design and instruction that position historically marginalized students as constructors of mathematical knowledge, thus promoting their development of positive social and mathematical identities.

These findings further contextualize exemplary practices detailed in this guide. Teaching practices have a significant impact on students' learning experiences and outcomes but are only part of the story. Awareness of the impact on students' identities and broader institutional issues can prompt instructors to adhere to the core principles of evidence-based practices and the inequities they aim to correct. Equity is a process, not an end goal.

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