The 2017 Putnam Competition Problems and Solutions

A1. Let S be the smallest set of positive integers such that

a) 2 is in S,
b) n is in S whenever n^2 is in S, and
c) $(n+5)^2$ is in S whenever n is in S.

Which positive integers are not in S?

(The set S is "smallest" in the sense that S is contained in any other such set.)

Answer. The positive integers that are not in S are 1 and the multiples of 5.

Solution. First note that by combining conditions c) and b), $n \in S$ implies $n+5 \in S$. Also, because $2 \in S$, we have $7^2 = 49 \in S$ and therefore $(49+5)^2 = 54^2 \in S$. Thus, because $54^2 \equiv 1 \pmod 5$, all sufficiently large positive integers that are $\equiv 1 \pmod 5$ are in S.

Now let $a > 1$ be an integer that is not a multiple of 5. Then the sequence $a, a^2, a^4, a^8, a^{16}, \ldots$ grows without bound, and by Fermat's little theorem (or by a check of cases mod 5) all the terms of the sequence starting with a^4 are $\equiv 1 \pmod 5$. Thus the sequence contains elements of S, and by repeated application of condition b) it follows that $a \in S$.

On the other hand, it is easy to check that the set of all integers greater than 1 that are not multiples of 5 satisfies conditions a), b), and c), so this is the set S, and the complement of S consists of the integers listed in the answer.

A2. Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

Solution 1. Let a sequence of polynomials be defined by $P_{-1}(x) = 0, P_0(x) = 1$, and $P_n(x) = xP_{n-1}(x) - P_{n-2}(x)$ for all $n \geq 1$. Clearly, these polynomials have integer coefficients, so it will be enough to show that $Q_n(x) = P_n(x)$ for all $n \geq 0$. Note that for all $n \geq 2$,

$$\begin{aligned}
\begin{pmatrix} P_n(x) & -P_{n-1}(x) \\ P_{n-1}(x) & -P_{n-2}(x) \end{pmatrix} &= \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} P_{n-1}(x) & -P_{n-2}(x) \\ P_{n-2}(x) & -P_{n-3}(x) \end{pmatrix} = \cdots \\
&= \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} P_1(x) & -P_0(x) \\ P_0(x) & -P_{-1}(x) \end{pmatrix} \\
&= \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}^{n}.
\end{aligned}$$

Taking determinants of both sides, we see that

$$-P_n(x)P_{n-2}(x) + (P_{n-1}(x))^2 = 1^n = 1, \quad \text{which implies} \quad P_n(x) = \frac{(P_{n-1}(x))^2 - 1}{P_{n-2}(x)}.$$

But this is precisely the defining recurrence relation for $Q_n(x)$, and since $P_0(x) = Q_0(x)$ and $P_1(x) = Q_1(x)$, we are done.

Solution 2. To show that the $Q_n(x)$ are polynomials with integer coefficients, we will show that they also satisfy the simpler recurrence $Q_n(x) = x\,Q_{n-1}(x) - Q_{n-2}(x)$. The proof is by induction on $n \geq 2$; for $n = 2$ we check directly that

$$Q_2(x) = \frac{(Q_1(x))^2 - 1}{Q_0(x)} = x^2 - 1 = x\,Q_1(x) - Q_0(x).$$

Assuming that $Q_n(x) = x\,Q_{n-1}(x) - Q_{n-2}(x)$ for $n = N-1$, define $R_N(x) = x\,Q_{N-1}(x) - Q_{N-2}(x)$. Then

$$\begin{aligned}
R_N(x)Q_{N-2}(x) &= xQ_{N-1}(x)Q_{N-2}(x) - (Q_{N-2}(x))^2 \\
&= xQ_{N-1}(x)Q_{N-2}(x) - \big(\frac{(Q_{N-2}(x))^2 - 1}{Q_{N-3}(x)} \cdot Q_{N-3}(x) + 1\big) \\
&= xQ_{N-1}(x)Q_{N-2}(x) - Q_{N-1}(x)Q_{N-3}(x) - 1 \\
&= Q_{N-1}(x)\big(xQ_{N-2}(x) - Q_{N-3}(x)\big) - 1 \\
&= (Q_{N-1}(x))^2 - 1 \quad \text{by induction hypothesis, so} \\
R_N(x) &= \frac{(Q_{N-1}(x))^2 - 1}{Q_{N-2}(x)} = Q_N(x), \text{ and we are done.}
\end{aligned}$$

A3. Let a and b be real numbers with $a < b$, and let f and g be continuous functions from $[a, b]$ to $(0, \infty)$ such that $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$ but $f \neq g$. For every positive integer n, define

$$I_n = \int_a^b \frac{(f(x))^{n+1}}{(g(x))^n}\,dx\,.$$

Show that $I_1, I_2, I_3, \ldots$ is an increasing sequence with $\lim_{n\to\infty} I_n = \infty$.

Solution. First consider

$$\begin{aligned}
I_n - I_{n-1} &= \int_a^b \frac{(f(x))^{n+1}}{(g(x))^n}\,dx - \int_a^b \frac{(f(x))^n}{(g(x))^{n-1}}\,dx \\
&= \int_a^b \frac{(f(x))^{n+1} - g(x)(f(x))^n}{(g(x))^n}\,dx \\
&= \int_a^b \frac{\big((f(x))^n - (g(x))^n\big)\big(f(x) - g(x)\big)}{(g(x))^n} + \big(f(x) - g(x)\big)\,dx \\
&= \int_a^b \frac{\big((f(x))^n - (g(x))^n\big)\big(f(x) - g(x)\big)}{(g(x))^n}\,dx + \int_a^b \big(f(x) - g(x)\big)\,dx.
\end{aligned}$$

The first integral is positive, because the integrand is nonnegative, continuous, and not everywhere zero (given $f \neq g$). The second integral is zero, because the integrals of f and g over the interval are equal. Thus $I_n - I_{n-1}$ is positive for all $n \geq 2$, so the sequence (I_n) is increasing.

Next, we claim that there exist a subinterval $I \subseteq [a, b]$ of positive length L and a constant $M > 1$ so that $\dfrac{f(x)}{g(x)} \geq M$ for all $x \in I$. Proof: Because $f(x) \neq g(x)$ for some $x \in [a, b]$ and $f - g$ is continuous, there is a subinterval on which either $f - g > 0$ or $f - g < 0$. But in the latter case there must also be a point $x \in [a, b]$ (and hence a subinterval of $[a, b]$) where $f(x) > g(x)$, otherwise we would have

$\int_a^b f(x)\ dx < \int_a^b g(x)\ dx$. Now we can take any closed subinterval I on which $f(x) > g(x)$, and we can take M to be the minimum value of the continuous function $\dfrac{f(x)}{g(x)}$ on that subinterval.

Finally,

$$\begin{aligned} I_n = \int_a^b \frac{(f(x))^{n+1}}{(g(x))^n}\ dx &\geq \int_I \frac{(f(x))^{n+1}}{(g(x))^n}\ dx \\ &= \int_I \left(\frac{f(x)}{g(x)}\right)^n f(x)\ dx \\ &\geq M^n \int_I f(x)\ dx\,. \end{aligned}$$

Because $M > 1$ and $\int_I f(x)\ dx$ is a positive constant, this shows that $\lim_{n\to\infty} I_n = \infty$.

A4. A class with $2N$ students took a quiz, on which the possible scores were $0, 1, \ldots, 10$. Each of these scores occurred at least once, and the average score was exactly 7.4. Show that the class can be divided into two groups of N students in such a way that the average score for each group was exactly 7.4.

Solution 1. Let the student scores in non-decreasing order be

$$0 = s_1 \leq s_2 \leq \cdots \leq s_{2N} = 10,$$

and let the sum of all the scores be S. From the given average score, we have $S = (2N)(7.4) = \dfrac{74N}{5}$, so N is divisible by 5 and S is even. Let $a_k = s_{2k} - s_{2k-1}$; because all scores occur, each a_k must be either 0 or 1. Now let $t = a_1 + a_2 + \cdots + a_N$, and note that t has the same parity as S, so t is also even. Then there is some n with $n < N$ for which $a_1 + a_2 + \cdots + a_n = \frac{1}{2}t$. Then for the group of N students whose scores are $s_2, s_4, \ldots, s_{2n}, s_{2n+1}, s_{2n+3}, \ldots, s_{2N-1}$, the sum of their scores is

$$\begin{aligned} &s_2 + s_4 + \cdots + s_{2n} + s_{2n+1} + \cdots + s_{2N-1} \\ &= (a_1 + s_1) + \cdots + (a_n + s_{2n-1}) + s_{2n+1} + \cdots + s_{2N-1} \\ &= (a_1 + \cdots + a_n) + (s_1 + s_3 + \cdots + s_{2N-1}) \\ &= \frac{1}{2}\big(t + 2s_1 + 2s_3 + \cdots + 2s_{2N-1}\big) \\ &= \frac{1}{2}\big((s_1 + s_3 + \cdots + s_{2N-1}) + ((a_1 + s_1) + \cdots + (a_N + s_{2N-1})\big) \\ &= \frac{1}{2}(s_1 + s_3 + \cdots + s_{2N-1} + s_2 + \cdots + s_{2N}) = \frac{1}{2}S = N(7.4), \end{aligned}$$

so the average score for this group is 7.4 (and thus the average score for the complementary group is also 7.4), as desired.

Solution 2. As in the first solution, the number of students is divisible by 10; say the scores are $s_1, \ldots, s_{10m}$ and thus satisfy $s_1 + \cdots + s_{10m} = 74m$. If it is possible to rearrange the scores such that $s_1 + \cdots + s_{5m} = 37m$, we are done; otherwise, rearrange them so that

$$S_1 = s_1 + \cdots + s_{5m} = 37m - \delta, \quad S_2 = s_{5m+1} + \cdots + s_{10m} = 37m + \delta$$

with $\delta > 0$ as small as possible. If a term in S_1 is exactly one less than a term in S_2, then we can exchange those terms and make δ smaller, contradiction. So because all scores occur, if the smallest term in S_1 is a, then $a+1$ must also appear in S_1, and repeating this argument, $a+1, a+2, \ldots$ must all appear in S_1. Let the largest term in S_2 be b. Then $5ma \le S_1 < S_2 \le 5mb$, so $b > a$, so $b-1$ must appear in S_1 and we can exchange $b-1$ from S_1 and b from S_2 after all to reduce δ, contradiction.

A5. Each of the integers from 1 to n is written on a separate card, and then the cards are combined into a deck and shuffled. Three players, A, B, and C, take turns in the order $A, B, C, A, \ldots$ choosing one card at random from the deck. (Each card in the deck is equally likely to be chosen.) After a card is chosen, that card and all higher-numbered cards are removed from the deck, and the remaining cards are reshuffled before the next turn. Play continues until one of the three players wins the game by drawing the card numbered 1.

Show that for each of the three players, there are arbitrarily large values of n for which that player has the highest probability among the three players of winning the game.

Solution. For every positive integer n, let A_n, B_n, C_n denote the probabilities that players A, B, C (respectively) win the game for that value of n. Note that $A_1 = 1$, $B_1 = C_1 = 0$ (if $n = 1$, there is only one card, and A gets to choose it). For $n > 1$, if player A chooses the card numbered k with $k > 1$, the game then proceeds like the original game with $k-1$ cards, except that B now chooses first, C chooses second, and A takes on C's original role, choosing third. Therefore, we have the recurrence relations

$$\begin{aligned} A_n &= \frac{1}{n} + \frac{1}{n}\,C_1 + \frac{1}{n}\,C_2 + \cdots + \frac{1}{n}\,C_{n-1}\,, \\ B_n &= \frac{1}{n}\,A_1 + \frac{1}{n}\,A_2 + \cdots + \frac{1}{n}\,A_{n-1}\,, \\ C_n &= \frac{1}{n}\,B_1 + \frac{1}{n}\,B_2 + \cdots + \frac{1}{n}\,B_{n-1}\,. \end{aligned}$$

Multiplying through by n and then subtracting the equations for n from those for $n+1$ yields

$$(n+1)\,A_{n+1} - n\,A_n = C_n\,, \quad (n+1)\,B_{n+1} - n\,B_n = A_n\,, \quad (n+1)\,C_{n+1} - n\,C_n = B_n\,.$$

Thus we have

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \\ C_{n+1} \end{pmatrix} = M_n \begin{pmatrix} A_n \\ B_n \\ C_n \end{pmatrix}, \quad \text{where} \quad M_n = \frac{1}{n+1}\begin{pmatrix} n & 0 & 1 \\ 1 & n & 0 \\ 0 & 1 & n \end{pmatrix}.$$

The eigenvalues of the matrix $\begin{pmatrix} n & 0 & 1 \\ 1 & n & 0 \\ 0 & 1 & n \end{pmatrix}$ are the roots of $(n-\lambda)^3 + 1 = 0$, so if we let $\omega = e^{2\pi i/3}$, they are given by $n - \lambda = -1$, $n - \lambda = -\omega$, $n - \lambda = -\omega^2$. Dividing by

$n+1$, we find the eigenvalues

$$1, \quad \frac{n+\omega}{n+1}, \quad \frac{n+\omega^2}{n+1}$$

of M_n, and a straightforward computation yields corresponding eigenvectors

$$\begin{pmatrix}1\\1\\1\end{pmatrix}, \quad \begin{pmatrix}1\\\omega^2\\\omega\end{pmatrix}, \quad \begin{pmatrix}1\\\omega\\\omega^2\end{pmatrix}$$

respectively. In particular, the eigenvectors are the same for each n, and so we can use them, together with

$$\begin{pmatrix}A_{n+1}\\B_{n+1}\\C_{n+1}\end{pmatrix} = M_n \begin{pmatrix}A_n\\B_n\\C_n\end{pmatrix},$$

to find expressions for the probabilities A_n, B_n, C_n, as follows:

$$\begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\\omega^2\\\omega\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\\omega\\\omega^2\end{pmatrix}, \quad \text{so}$$

$$\begin{aligned}
\begin{pmatrix}A_n\\B_n\\C_n\end{pmatrix} &= M_{n-1}M_{n-2}\cdots M_1 \begin{pmatrix}1\\0\\0\end{pmatrix} \\
&= \frac{1}{3} M_{n-1}M_{n-2}\cdots M_1 \begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{1}{3} M_{n-1}M_{n-2}\cdots M_1 \begin{pmatrix}1\\\omega^2\\\omega\end{pmatrix} \\
&\qquad + \frac{1}{3} M_{n-1}M_{n-2}\cdots M_1 \begin{pmatrix}1\\\omega\\\omega^2\end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{1}{3}\prod_{k=1}^{n-1}\frac{k+\omega}{k+1}\cdot\begin{pmatrix}1\\\omega^2\\\omega\end{pmatrix} + \frac{1}{3}\prod_{k=1}^{n-1}\frac{k+\omega^2}{k+1}\cdot\begin{pmatrix}1\\\omega\\\omega^2\end{pmatrix}.
\end{aligned}$$

Let $P = \prod_{k=1}^{n-1}\frac{k+\omega}{k+1}$. Then, because $\omega^2 = \overline{\omega}$, we have

$$\begin{aligned}
A_n &= \frac{1}{3}(1+P+\overline{P}) = \frac{1}{3}(1+2\,\mathrm{Re}(P)), \\
B_n &= \frac{1}{3}(1+P\overline{\omega}+\overline{P}\omega) = \frac{1}{3}(1+2\,\mathrm{Re}(P\overline{\omega})), \\
C_n &= \frac{1}{3}(1+P\omega+\overline{P\omega}) = \frac{1}{3}(1+2\,\mathrm{Re}(P\omega)).
\end{aligned}$$

Thus, which of the three players has the highest probability of winning the game depends only on which of $\mathrm{Re}(P)$, $\mathrm{Re}(P\overline{\omega})$, $\mathrm{Re}(P\omega)$ is the largest; that, in turn, depends only on the argument of P. Specifically, A_n is largest when $\mathrm{Arg}(P)$ is in the interval $[-\pi/3, \pi/3]$, B_n is largest when $\mathrm{Arg}(P)$ is in $[\pi/3, \pi]$, and C_n is largest when $\mathrm{Arg}(P)$

is in $[-\pi, -\pi/3]$. But modulo 2π we have

$$\operatorname{Arg}(P) = \sum_{k=1}^{n-1} \operatorname{Arg}\Big(\frac{k+\omega}{k+1}\Big) = \sum_{k=1}^{n-1} \arctan \frac{\sqrt{3}}{2k-1}\,.$$

Because $\arctan x \sim x$ as $x \to 0$, we have

$$\arctan \frac{\sqrt{3}}{2k-1} \sim \frac{\sqrt{3}/2}{k} \quad \text{as } k \to \infty,$$

so the values of $\arg(P)$ above are the partial sums of a divergent series, but the difference between successive partial sums tends to 0. Thus modulo 2π, the values will revisit each of the three intervals $[-\pi/3, \pi/3]$, $[\pi/3, \pi]$, $[-\pi, -\pi/3]$ infinitely often, which concludes the proof.

A6. The 30 edges of an icosahedron are distinguished by labeling them $1, 2, \ldots, 30$. How many different ways are there to paint each edge red, white, or blue such that each of the 20 triangular faces of the icosahedron has two edges of the same color and a third edge of a different color?

Answer. $2^{20} \cdot 3^{10} = 12^{10} = 61,917,364,224$.

Solution. Let $F = \mathbb{F}_3 = \{0, 1, 2\}$ be the field with 3 elements; let those elements correspond to "red", "white", and "blue" respectively. Then a way of painting the edges can be represented by an element of the vector space $V = F^{30}$. Label the 20 faces of the icosahedron $1, 2, \ldots, 20$, and for the i-th face, define a function $f_i : V \to F$ by

$$f_i(v) = \text{the sum (modulo 3) of the colors in } v \text{ of the edges of face } i\,.$$

Using these functions, we can define a linear transformation $T : V \to F^{20}$ by

$$T(v) = (f_1(v), f_2(v), \ldots, f_{20}(v)).$$

The condition that face i has two edges of the same color and a third edge of a different color is easily checked to be equivalent to $f_i(v) \neq 0$, so we are looking for the cardinality of the inverse image under T of the subset $\{1, 2\}^{20}$ of F^{20}.

We can show that T is surjective by showing that each of the standard basis vectors of F^{20} is in the image, that is, that for any given face we can find a way of coloring the edges of the icosahedron so that the sum of the colors around that face is 1 and the sum of the colors around any other face is 0. By symmetry, it is enough to do this for a single face. One specific way to achieve this is as follows: Pick any vertex v of the icosahedron and color the five edges emanating from that vertex, say to the vertices v_1, v_2, v_3, v_4, v_5, with the colors $2, 1, 2, 1, 1$ in order. Color all other edges of the icosahedron with color 0, except for the edge connecting v_4 and v_5 which gets color 1. Then around every face adjacent to v, the sum of the colors is either $1+2+0$ or $1+1+1$, so 0 in either case, and with one exception, around every face not adjacent to v all the colors are 0 and so their sum is certainly 0. The one exception is the face which is not adjacent to v but which does have $v_4 v_5$ among its edges; for that face, the sum of the colors is 1.

Now that we know T is surjective, we can get the desired cardinality by multiplying the size of the subset $\{1, 2\}^{20}$, which is 2^{20}, by the size of $\ker(T)$. But we know that

$\dim \ker(T) + \dim \operatorname{im}(T) = \dim V = 30$, so $\dim \ker(T) = 30 - 20 = 10$ and $\ker(T)$ has 3^{10} elements; the answer follows.

(The B section starts on the next page.)

B1. Let L_1 and L_2 be distinct lines in the plane. Prove that L_1 and L_2 intersect if and only if, for every real number $\lambda \neq 0$ and every point P not on L_1 or L_2, there exist points A_1 on L_1 and A_2 on L_2 such that $\overrightarrow{PA_2} = \lambda\,\overrightarrow{PA_1}$.

Solution 1. To show "if", take $\lambda = 1$ and any point P not on L_1 or L_2. Then $\overrightarrow{PA_2} = \lambda\,\overrightarrow{PA_1}$ implies that L_1 and L_2 intersect at $A_1 = A_2$. To show "only if", assume L_1 and L_2 have the directions of the vectors $\mathbf{v_1} =< a_1, b_1 >, \mathbf{v_2} =< a_2, b_2 >$ respectively and intersect at the point (x_0, y_0). Then the line L_i has parametric equations $x = x_0 + a_i t, y = y_0 + b_i t$; let $P = (p, q)$ be a point that is not on either line. Then there are points A_1, A_2 on L_1, L_2 respectively with $\overrightarrow{PA_2} = \lambda\,\overrightarrow{PA_1}$ if and only if there are real numbers t_1, t_2 (the parameter values for A_1, A_2) such that

$$x_0 + a_2 t_2 - p = \lambda(x_0 + a_1 t_1 - p) \quad \text{and} \quad y_0 + b_2 t_2 - q = \lambda(y_0 + b_1 t_1 - q).$$

This can be written as a system of two linear equations in t_1, t_2 whose coefficient matrix is

$$\begin{pmatrix} \lambda a_1 & -a_2 \\ \lambda b_1 & -b_2 \end{pmatrix};$$

because $\lambda \neq 0$ and the vectors $\mathbf{v_1}, \mathbf{v_2}$ are linearly independent (because the lines are not parallel), this matrix has nonzero determinant, so it is invertible and thus there is a unique solution for t_1, t_2.

Comment. This proof actually shows that the condition "P not on L_1 or L_2" is superfluous. In fact, if P is on L_i, then the point A_{3-i} on the other line will always be the intersection point (x_0, y_0), whereas the point A_i varies with λ - unless P itself is the intersection point, in which case $A_1 = A_2 = P$ for every λ.

Solution 2. To show "if", proceed as in the first solution. To show "only if", assume L_1 and L_2 intersect at a point Q, and take a point P that is not on L_1 or L_2. Let M_1, M_2 be the lines through P parallel to L_1, L_2 respectively; note that these lines divide the plane into four "quadrants", one of which contains the intersection point Q. Now consider a variable line K that passes through P. Except when $K = M_1$ and when $K = M_2$, the line K will intersect the lines L_1, L_2 in points A_1, A_2, and there will be a nonzero real number $\lambda(K)$ such that $\overrightarrow{PA_2} = \lambda(K) \cdot \overrightarrow{PA_1}$. As the line K rotates around the point P, the number $\lambda(K)$ will vary continuously. It will be positive when the line K passes through the "quadrant" containing Q, approaching 0 as K approaches the line M_1 (so that the intersection point A_1 approaches infinity) and ∞ as K approaches M_2. Similarly, $\lambda(K)$ will be negative when the line K does not pass through the "quadrant" containing Q, approaching 0 as K approaches M_1 and $-\infty$ as K approaches M_2. By the intermediate value theorem, it follows that $\lambda(K)$ takes on all real values, and the claim follows.

B2. Suppose that a positive integer N can be expressed as the sum of k consecutive positive integers

$$N = a + (a+1) + (a+2) + \cdots + (a+k-1)$$

for $k = 2017$ but for no other values of $k > 1$. Considering all positive integers N with this property, what is the smallest positive integer a that occurs in any of these expressions?

Answer. $a = 16$.

Solution. If we do have $N = a + (a + 1) + (a + 2) + \cdots + (a + k - 1)$ with an integer $a > 0$, then we have $N = \frac{k}{2}(2a + k - 1)$. In particular, for $k = 2017$ we have $N = 2017(a + 1008)$. Now suppose $a + 1008$ has some odd factor $s > 1$, so $N = s(2017t)$, say. Then we can write N as the sum of s consecutive integers $\cdots + (2017t-1) + 2017t + (2017t+1) + \cdots$, and these are all positive unless $s > 2017t$. So if $a + 1008$ has an odd factor $s > 1$, then $s \geq 2017$ and thus $N \geq 2017^2$.

If, on the other hand, $a+1008$ has no odd factor > 1, then $a+1008$ is a power of 2; the smallest possibility for this power is $2^{10} = 1024$, and this gives $N = 2017 \cdot 1024$, which is less than 2017^2. We now show that this integer does have the specified property. If we have $2017 \cdot 1024 = a+(a+1)+(a+2)+\cdots+(a+k-1) = \frac{k}{2}(2a+k-1)$, then $k(2a+k-1) = 2017 \cdot 2^{11}$, and because 2017 is prime, exactly one of k and $2a+k-1$ is divisible by 2017 and the other factor is a power of 2. If k is greater than 2017 and divisible by 2017, then k also has at least one factor 2, so $2a + k - 1$ is odd and greater than 1, so not a power of 2, contradiction. If $2a + k - 1$ is divisible by 2017, then $k > 1$ is a power of 2, so $2a + k - 1$ is odd, so $2a + k - 1 = 2017$, but then $k \leq 2017$ and so $k \leq 2^{10} = 1024$, $k(2a + k - 1) \leq 2017 \cdot 2^{10}$, contradiction. Thus the only possible $k > 1$ is $k = 2017$, for which $N = 2017 \cdot 1024$ is the sum of 2017 consecutive integers starting with 16. Because this value of N is less than 2017^2, it is the smallest possibility, and so the corresponding value $a = 16$ is also the smallest one that occurs.

B3. Suppose that $f(x) = \sum_{i=0}^{\infty} c_i x^i$ is a power series for which each coefficient c_i is 0 or 1. Show that if $f(2/3) = 3/2$, then $f(1/2)$ must be irrational.

Solution. Note that by the comparison test with the geometric series $\sum_{i=0}^{\infty} |x|^i$, the power series $f(x)$ is absolutely convergent for $x \in (-1, 1)$; in particular, $f(1/2)$ is a real number. Suppose that $f(1/2)$ is rational. Because $f(1/2) = \sum_{i=0}^{\infty} c_i(\frac{1}{2})^i$, the binary expansion of this number reads $c_0.c_1c_2 \cdots c_i \cdots$, and (by the same argument as for decimal expansions) the sequence $c_0, c_1, \ldots$ of its binary digits is eventually periodic. If that sequence has eventual period k, then $f(x)$ can be written in the form

$$\begin{aligned} f(x) &= p_1(x) + p_2(x)(1 + x^k + x^{2k} + x^{3k} + \cdots) \\ &= p_1(x) + \frac{p_2(x)}{1 - x^k} \\ &= \frac{p(x)}{1 - x^k} \end{aligned}$$

for some polynomials $p_1(x), p_2(x), p(x)$ with integer coefficients (in fact, $p_1(x)$ and $p_2(x)$ have coefficients in $\{0, 1\}$). Then we have

$$\frac{3}{2} = f(\frac{2}{3}) = \frac{p(\frac{2}{3})}{1 - (\frac{2}{3})^k} = \frac{3^k\, p(\frac{2}{3})}{3^k - 2^k}\,.$$

However, the denominator of the fraction on the right is odd, which is a contradiction; it follows that $f(1/2)$ must be irrational.

B4. Evaluate the sum

$$\sum_{k=0}^{\infty}\left(3\cdot\frac{\ln(4k+2)}{4k+2}-\frac{\ln(4k+3)}{4k+3}-\frac{\ln(4k+4)}{4k+4}-\frac{\ln(4k+5)}{4k+5}\right)=$$

$$3\cdot\frac{\ln 2}{2}-\frac{\ln 3}{3}-\frac{\ln 4}{4}-\frac{\ln 5}{5}+3\cdot\frac{\ln 6}{6}-\frac{\ln 7}{7}-\frac{\ln 8}{8}-\frac{\ln 9}{9}+3\cdot\frac{\ln 10}{10}-\cdots.$$

(As usual, $\ln x$ denotes the natural logarithm of x.)

Answer. $\ln^2(2)$.

Solution. First consider the series

$$\frac{\ln 1}{1}-\frac{\ln 2}{2}+\frac{\ln 3}{3}-\frac{\ln 4}{4}+\cdots,$$

which converges by the alternating series test. (Details: $\lim\limits_{n\to\infty}\dfrac{\ln n}{n}=0$, the sequence $\left(\dfrac{\ln n}{n}\right)$ is decreasing for $n\geq 3$ because $\frac{d}{dx}\left(\frac{\ln x}{x}\right)=\frac{1-\ln x}{x^2}<0$ for $x>e$, and after the first term which is zero, the terms alternate in sign.) Let a be the sum of this series, and let S be the desired sum. Then

$$\begin{aligned}
S+a &= 2\cdot\frac{\ln 2}{2}-2\cdot\frac{\ln 4}{4}+2\cdot\frac{\ln 6}{6}-2\cdot\frac{\ln 8}{8}+2\cdot\frac{\ln 10}{10}-\cdots\\
&=\frac{\ln 2}{1}-\frac{\ln 2+\ln 2}{2}+\frac{\ln 2+\ln 3}{3}-\frac{\ln 2+\ln 4}{4}+\frac{\ln 2+\ln 5}{5}-\cdots\\
&=\ln 2\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\right)+a\\
&=\ln 2\,(\ln 2)+a\,,
\end{aligned}$$

and it follows that $S=\ln^2(2)$, as claimed.

B5. A line in the plane of a triangle T is called an *equalizer* if it divides T into two regions having equal area and equal perimeter. Find positive integers $a>b>c$, with a as small as possible, such that there exists a triangle with side lengths a, b, c that has exactly two distinct equalizers.

Answer. $a=9, b=8, c=7$.

Solution. As usual, let A be the vertex of the triangle opposite the side of length a, etc. We first consider the case that an equalizer separates vertex A from (the longest) side BC. Then the equalizer intersects side AB at a point Y and side AC at a point Z; say AY has length λc and AZ has length μb, so $\lambda,\mu\in[0,1]$. Then if θ is the angle at vertex A, the part of T on the same side of the equalizer as A is a triangle with area

$$\frac{1}{2}(\lambda c)(\mu b)\,\sin\theta=\lambda\mu\,\mathrm{Area}(T),$$

so $\lambda\mu=\frac{1}{2}$ and, in particular, $\lambda,\mu\in[\frac{1}{2},1]$. Meanwhile, the "equal perimeter" property of the equalizer implies that $\lambda c+\mu b=(1-\lambda)c+a+(1-\mu)b$, so $\lambda c+\mu b=\frac{a+b+c}{2}$. Multiplying by λ and then eliminating μ leads to the quadratic equation

$$c\lambda^2-\frac{a+b+c}{2}\lambda+\frac{b}{2}=0\,.$$

If we denote the polynomial on the left-hand side of this equation by $p(\lambda)$, then $p(\frac{1}{2}) = \frac{b-a}{4} < 0$ and $p(1) = \frac{c-a}{2} < 0$, so the entire interval $[\frac{1}{2}, 1]$ is between the roots of the quadratic polynomial (which is positive for large $|\lambda|$). This shows that, in fact, no such equalizer is possible: Any equalizer must intersect the longest side of the triangle, BC.

Now suppose that an equalizer separates vertex B from side AC. If the side lengths of the parts of the sides of BA, BC that are on the same side of the equalizer as B are $\lambda c, \mu a$ respectively, then we have, by an argument to the one in the previous case, $\lambda\mu = \frac{1}{2}$ and $\lambda c + \mu a = (1-\lambda)c + b + (1-\mu)a$, from which we get

$$p(\lambda) = c\lambda^2 - \frac{a+b+c}{2}\lambda + \frac{a}{2} = 0\,.$$

This time, $p(\frac{1}{2}) = \frac{a-b}{4} > 0$ while $p(1) = \frac{c-b}{2} < 0$, so there will be exactly one root of $p(\lambda)$ in the interval $[\frac{1}{2}, 1]$, and thus there will be exactly one equalizer that intersects both the shortest and the longest side of T.

Finally, consider equalizers that separate C from the shortest side, AB, of the triangle. If the usual side lengths are $\lambda b, \mu a$, we get, after a similar calculation,

$$p(\lambda) = b\lambda^2 - \frac{a+b+c}{2}\lambda + \frac{a}{2} = 0\,.$$

Now we have $p(\frac{1}{2}) = \frac{a-c}{4} > 0$ and $p(1) = \frac{b-c}{2} > 0$, so $p(\lambda)$ can have either no roots, two roots, or a double root in the interval $[\frac{1}{2}, 1]$. Because we have exactly one equalizer from an earlier case, there will be two distinct equalizers in all for T if and only if $p(\lambda)$ has a double root. This happens exactly when

$$\left(\frac{a+b+c}{2}\right)^2 = 4 \cdot b \cdot \frac{a}{2}, \quad \text{that is, when} \quad (a+b+c)^2 = 8ab\,.$$

So we are looking for a solution of this Diophantine equation in positive integers, with $a > b > c$ and a as small as possible. In particular, we may assume that a and b are relatively prime, because any common prime factor they have would also divide c and then the entire solution could be scaled down by that factor. Then because $8ab$ is a perfect square, there are positive integers m and n so that either $a = 2m^2, b = n^2$ or $a = m^2, b = 2n^2$. In the first case b is odd (because a, b are relatively prime), so because $b > c$ is a perfect square we have $b \geq 3^2 = 9$, so $a > 9$. In the second case a is odd, so $a \geq 9$, so the smallest value of a that might be possible is 9. Trying this case, we get $(9+b+c)^2 = 72b$; b is now of the form $2n^2$ and less than 9, so $b = 2$ or $b = 8$. For $b = 2$ we would have $(11+c)^2 = 144$, so $c = 1$, but there is no triangle with side lengths $9, 2, 1$. However, for $b = 8$ we get $(17+c)^2 = 72 \cdot 8$, which yields $c = 7$. There is a triangle with side lengths $9, 8, 7$, and we have found the answer.

B6. Find the number of ordered 64-tuples $(x_0, x_1, \ldots, x_{63})$ such that $x_0, x_1, \ldots, x_{63}$ are distinct elements of $\{1, 2, \ldots, 2017\}$ and

$$x_0 + x_1 + 2x_2 + 3x_3 + \cdots + 63x_{63}$$

is divisible by 2017.

Answer. $\dfrac{2016!}{1953!} - 2016 \cdot 63!$.

Solution. Note that the sum of the coefficients in $x_0 + x_1 + 2x_2 + 3x_3 + \cdots + 63x_{63}$ is

$$1 + (1 + 2 + \cdots + 63) = 1 + 63 \cdot 32 = 2017.$$

Because 2017 is prime, we can think of the x_i as elements of the field $\mathbb{F}_{2017}$, and we then have a special case of the following problem:

Given positive integers $a_1, a_2, \ldots, a_k$ with $a_1 + a_2 + \cdots + a_k = 2017$, find the number of solutions of the equation $a_1x_1 + a_2x_2 + \cdots + a_kx_k = 0$ in the field $\mathbb{F}_{2017}$ for which $x_1, x_2, \ldots, x_k$ are distinct.

We will show, by induction on k, that the answer depends only on k and is

$$f(k) = \frac{2016!}{(2017-k)!} - (-1)^k \cdot 2016 \cdot (k-1)!\,.$$

In our particular case, we have $k = 64$, leading to the numerical answer given above.

For the base case $k = 1$, the only possible equation is $2017\,x_1 = 0$, which has 2017 solutions in the field, and we see that $f(1) = 1 + 2016$ does give the number of solutions. For $k > 1$, first note that because $a_1, \ldots, a_k$ are all positive and add to 2017, a_k cannot be zero in the field. Therefore, given any distinct $x_1, \ldots, x_{k-1}$ in the field $\mathbb{F}_{2017}$, there will be a unique choice of x_k for which $(x_1, x_2, \ldots, x_k)$ is a solution; however, that x_k may not be distinct from all of $x_1, \ldots, x_{k-1}$.

The number of choices for an ordered $(k-1)$-tuple $(x_1, \ldots, x_{k-1})$ of distinct elements from $\mathbb{F}_{2017}$ is $2017 \cdot 2016 \cdots (2019-k) = \dfrac{2017!}{(2018-k)!}$. So to get the answer, we need to subtract from this the number of solutions to $a_1x_1 + a_2x_2 + \cdots + a_kx_k = 0$ for which $x_1, \ldots, x_{k-1}$ are distinct but x_k is equal to one of $x_1, \ldots, x_{k-1}$. If, in fact, x_k is equal to x_i, then we have a solution with distinct $x_1, \ldots, x_{k-1}$ to the equation $a_1x_1 + a_2x_2 + \cdots + (a_i + a_k)x_i + \cdots + a_{k-1}x_{k-1} = 0$. But the coefficients of this equation are positive integers that add to 2017, so by induction hypothesis there are $f(k-1)$ such solutions. Because i can be any of the numbers $1, \ldots, k-1$, we can conclude that the total number of solutions to $a_1x_1 + a_2x_2 + \cdots + a_kx_k = 0$ with distinct $x_1, x_2, \ldots, x_k \in \mathbb{F}_{2017}$ is

$$\begin{aligned}
&\frac{2017!}{(2018-k)!} - (k-1)f(k-1) = \\
&\qquad = \frac{2017!}{(2018-k)!} - (k-1)\left(\frac{2016!}{(2018-k)!} - (-1)^{k-1} \cdot 2016 \cdot (k-2)!\right) \\
&\qquad = \frac{(2018-k) \cdot 2016!}{(2018-k)!} - (-1)^k \cdot 2016 \cdot (k-1)! \\
&\qquad = \frac{2016!}{(2017-k)!} - (-1)^k \cdot 2016 \cdot (k-1)! \\
&\qquad = f(k), \quad \text{completing the induction.}
\end{aligned}$$