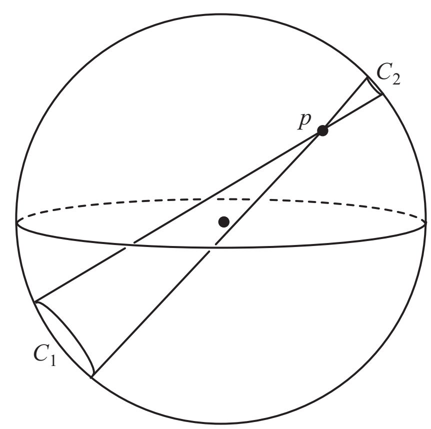
George Pólya Awards

Peter McGrath

"Newton's shell theorem via Archimedes' hat box and single variable calculus," *College Mathematics Journal*, 49:2, 109–113, 10.1080/07468342.2018.1411655.

This lovely short paper revisits some classical theorems in a lively fashion. The first problem addressed is the following: if an orange is covered by chocolate and you are offered the chance to slice the orange with two parallel cuts at distance d apart (thus creating a spherical segment), which cuts should you choose so as to maximize the area of chocolate on your piece? Some readers will recognize the Archimedes "hat box" theorem, which proves that all spherical segments with the same distance d have exactly the same area.

The author recalls briefly the modern proof with integral calculus and then discusses the argument given by Archimedes nearly two millennia before, which establishes that the area of the spherical segment is that same as that of its horizontal projection onto a cylinder tangent to the sphere. This result of Archimedes was rediscovered by Johann Heinrich Lambert in 1772 and is used in cartography: horizontal projection onto a cylinder allows the drawing of maps which preserve ratios (now called Lambert equivalent projection).



The author then moves to Newton's shell theorem. Consider a thin spherical shell of uniform mass density and total mass M. The net gravitational force exerted by the shell is zero at any point p inside the shell. As for a point p outside the shell, the gravitational force is the same as that exerted by a single mass point of mass M located at the center O of the spherical shell. Again, the author proposes a clever proof. First, consider the case where p is inside the shell, and consider a cone with vertex at p. This cone cuts two spherical shells. The squares of the distances from these shells to p are proportional to their areas, and hence to their masses. Thus their joint contribution exerts a zero gravitational force at p. The case where p is outside the sphere requires some computation, which the author elegantly carries out in the spirit of the hat box theorem. By a symmetry argument the force is directed along the line Op. Then this force is computed as the sum of the forces exerted by infinitely thin spherical segments perpendicular to Op, by means of a single variable elementary integral.

This paper is a gem, exhibiting the beauty of mathematical proof while at the same time revealing the mystery behind some deep classical results. It deserves to be used in calculus classes.

Response

I am delighted to receive the George Pólya Award. For many years, I have admired writings of several of the prior winners of this award, and it is a privilege to join their ranks. I would like to thank an anonymous referee, whose feedback and suggestions greatly improved the quality of the final article, as well as the editors of the *College Mathematics Journal*, for their support and dedication.

Biographical Sketch

Peter McGrath is currently a Hans Rademacher Instructor of Mathematics at the University of Pennsylvania and received his PhD in mathematics at Brown University in 2017. His research is in geometric analysis and primarily focuses on minimal surfaces.