Preface

> . . . it is extraordinary how fertile in properties the triangle is. Everyone can try his hand.
>
> *Treatise on the Arithmetical Triangle*, Blaise Pascal

> At the age of twenty-one, he wrote a treatise upon the binomial theorem, which has had a European vogue. On the strength of it he won the mathematical chair at one of our smaller universities, and had, to all appearances, a most brilliant career before him. (Sherlock Holmes about James Moriarty)
>
> *The Final Problem,* Sir Arthur Conan Doyle

> Who has anything new to say about the binomial theorem at this late date? At any rate, I am certainly not the man to know. (Moriarty to Watson)
>
> *The Seven-Per-Cent Solution,* Nicholas Meyer

> . . . every style of reasoning introduces a great many novelties, including new types of (i) objects, (ii) evidence, (iii) sentences, new ways of being a candidate for truth or falsehood, (iv) laws, or at any rate modalities, (v) possibilities. One will also notice, on occasion, new types of classification, and new types of explanations.
>
> Ian Hacking

According to the Oxford English Dictionary, a *pastiche* is a work that incorporates several different styles, drawn from a variety of sources, in the style of someone else. In this book I am consciously imitating the work *Exercices de Style* (Éditions Gallimard, Paris, 1947) of Raymond Queneau (1903–1976). In a page, Queneau introduces a banal story of a man on a bus that he then tells in 99 different ways. The manners of speech include philosophical, metaphorical, onomatopoeic, telegraphic; also the forms of an advertisement, a short play, a libretto, a tanka; in passive voice, imperfect tense; and so on. He celebrates the potential of language, and his experiments in style, even the most demanding ones to read, seek to delight the reader with new forms.

Queneau was no stranger to mathematics, having published a note in the *Comptes Rendus* and a paper in the *Journal of Combinatorial Theory* (both on s-additive sequences). He also published *Cent mille milliards de poèmes* (100,000,000,000,000 Poems[1]), consisting of a set of ten sonnets with a unified rhyme scheme from which a poem is constructed by making a choice of each line from among the corresponding lines in the ten sonnets. This scheme produces the 10^{14} distinct poems. The poet estimated that it would require more than 190 million years to recite them all.

Queneau was also the cofounder in 1960 of the experimental writing group OuLiPo (*Ouvroir de Littérature Potentielle*). From their earliest meetings the OuLiPo discussed the use of mathematical structures in literary endeavors. When strict rules (constraints) are chosen and followed, a writer needs to find new modes of expression. By focusing on new operations applied to literary works or generating new works, writing can become freer and more thorough, a laboratory of invention. The canonical example is *La Disparation* by George Perec (1936–1982), Denoël, Paris, 1969, a novel in which the vowel *e* has vanished. Such a text is called a *lipogram*.

Mathematicians and authors well known to the mathematical community have been members of the OuLiPo since its origin, including cofounder François Le Lionnais, Claude Berge, and Michèle Audin.

So what have I tried to do? I have chosen a (not at all) banal object in mathematics—the binomial coefficients—and written 99 short notes in which these numbers (or their cousins) play some role. Certain particular properties of these numbers appear in different exercises, seen from different viewpoints. The behavior of the binomial coefficients provides a framework that suggests properties of related families of numbers. The binomial coefficients even appear as a sort of character in a mathematical story, or they structure the choices in a piece. The temptation to write a mathematical pastiche of *Exercices de Style* has been on my mind for a long time, and it has also been taken up by others: Ludmilla Duchêne and Agnès Leblanc have written the delightful *Rationnel mon Q* (Hermann, Paris, 2010) in which 65 exercises explore the irrationality of the square root of 2.

What is *style*? According to Wikipedia, style is a manner of doing or presenting something. In mathematics, there may be an algebraic way to understand certain results, and there also may be a combinatorial way to understand the same fact. Thus viewpoint contributes to style. Other aspects of style might include the method of a proof, be it induction, recursion, or algorithm; or by contradiction, by the introduction of a fancy zero, or a conscious imitation of the proof of a different fact. On the other hand, the presentation of a mathematical proof might take a particular form of discourse.

[1] An English translation of *Cent mille milliards de poèmes* can be found in *Oulipo Compendium*, compiled by Harry Mathews and Alastair Brotchie. London: Atlas Press; Los Angeles: Make Now Press, 2005.

There is precious little discussion[2] of style in mathematics, even if mathematicians seem to have strong opinions. At the least, mathematicians seek to imitate their most admired authors' styles of discourse.

Like Moriarty I cannot hope to say anything new about the binomial coefficients. I am sure however that something new can be found on some pages by most readers. The exercises are intended to explore some of the possible ways to communicate mathematical ideas, and to reveal how binomial coefficients give mathematicians plenty to talk about. In some of the titles and in the *Style notes* at the end of the book, I try to identify the features of mathematical thinking, proving, and writing on which each piece is based. I am not immune to the enthusiasms of the OuLiPo. As a general constraint each piece is no longer than three pages. The communication is the thing, and I have tried to tell a little something in each exercise.

Mathematics is one of humanity's oldest and deepest arts. The challenge of communicating mathematical ideas and truths deserves some critical attention, if only for the love of it. Queneau intended his *Exercices de Style* to, in his words, "act as a kind of rust-remover to literature." I share the same intentions with him for my exercises. And, quoting Queneau,

> "If I have been able to contribute a little to this, then I am very proud, especially if I have done it without boring the reader too much."
>
> Raymond Queneau[3]

How to use this book

The book is too thin to be useful as a doorstop, and it will anchor few pages in the role of paperweight. I recommend reading bits at random. You will find that neighboring pages of any particular exercise may be related in topic and so, if that topic interests you, by all means explore. Of course, read with a pencil (or pen) at hand to scribble particular cases, missing steps, critical remarks, conjectures (the next Fermat?), comments for the author, or shopping lists, as the spirit moves you. If a fog of obfuscation surrounds any page, acquaint yourself with the *Style notes* found at the end of the exercises, containing helpful citations, some of the author's zany intentions, and valuable clues to hidden messages and

[2] See, for example, Variations du style mathématique by Claude Chevalley, *Revue de Métaphysique et de morale* **42**(1935) 375–384. A more extended discussion may be found in the article of Paolo Mancuso on mathematical style for the Stanford Encyclopedia of Philosophy, at the website: `plato.stanford.edu/entries/mathematical-style/`. Also, for a wide ranging discussion of style in mathematics see the notes from the Oberwolfach workshop, "*Disciplines and styles in pure mathematics*, 1800–2000", 28.II–6.III.2010, organized by David Rowe, Klaus Volkert, Phillippe Nabonnand, and Volker Remmert, found at `www.mfo.de/occasion/1009`.

[3] Quoted in *Many Subtle Channels* by Daniel Levin Becker, Harvard University Press, 2012.

inside jokes[4]. Prerequisites for the text are few. Most of the exercises can be read after a good high school mathematics education, some require an acquaintance with calculus, and a few are written in the language of undergraduate courses in analysis, algebra, even topology. I have marked the entries in the table of contents with a star $*$ if calculus is used, two stars if complex numbers, linear algebra or number theory is expected, and three stars if exposure to higher level courses might be helpful. Dear Reader, the styles of thinking and speaking in more advanced fields of mathematics are fascinating as well, and exercises at this level will give you a glimpse of what is ahead. You can check the *Style notes* for introductory texts in the relevant fields.

Enjoy the offerings here in the spirit of play in which they were written.

Acknowledgements

Thanks to Hamish Short who introduced me to Queneau's *Exercices de Style* and whose enthusiasm for it has infected me. Anastasia Stevens, Robert Ronan, and Lilian Zhao spent the summer of 2014 helping me find more ways in which the binomial coefficients appear in mathematics, providing subject matter for the exercises. Nancy Fox, Curtis Dozier, and Natalie Friedman shared their rhetorical expertise with me. Thanks also to folks who read bits, sometimes big bits, of the manuscript and offered their help in making it better: Ming-Wen An, Cristina Ballantine, Matthias Beck, Jan Cameron, Natalie Frank, Jerry Furey, Anthony Graves-McCleary, David Guichard, Liam Hunt, Beth Malmskog, Clemency Montelle, and Kim Plofker. The difficult work of the copy-editor was expertly done by Underwood Dudley whose prose I admire and whose advice improved this book. All remaining errors are mine. The editorial board of the MAA's Anneli Lax New Mathematical Library series, Timothy G. Feeman, Katharine Ott, Katherine Socha, James S. Tanton, and Jennifer M. Wilson, led by the ever-encouraging Karen Saxe, provided considerable wisdom and close readings of various versions of the manuscript. My thanks to all of you. The book is better for your time and effort. The efforts of Bev Ruedi to meet my typesetting demands were most appreciated. Thanks also to Steve Kennedy, the MAA's Acquisition Editor, who immediately got what I was trying to do.

[4] This sentence is a pangram—a sentence containing all of the letters of the alphabet.