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The Birth of the Meter

Randy K. Schwartz
Schoolcraft College, Livonia, MI

At first thought, it might seem like the selection of units for length and other physical quantities would be arbitrary—so how could it involve any significant mathematics?

But creating the world's first system of measurements that would *not* be arbitrary was one of the announced goals of the leaders of the French Revolution after they seized power in 1789. That is why they had to recruit some of the most eminent mathematicians of the time to work on the problem of defining the meter, which would be the new system's fundamental unit. The contentious and painful birth of the meter turned out to be an exciting chapter in the application of mathematics, a story that ties together seemingly disparate threads in the development of trigonometry, calculus, and differential equations.

In France in 1789, even the king saw the need for a new set of units to replace the roughly 250,000 different weights and measures that were being used locally in towns and rural areas. That confusing patchwork had caused countless disputes to erupt, whenever taxes were collected or goods were bought and sold.

But French scientists and legislators were vexed: how were they going to define the meter in a way that would reflect nature only, instead of a particular locality or culture? Even as revolutionary conflict and the Napoleonic wars raged in the 1790s, some of the most contentious debates carried out within the new French governing assemblies focused on this question.

Two camps emerged. One side argued for basing the definition of the meter on a pendulum clock, which would be calibrated in size so as to swing to and fro exactly once per second. The other side argued that the meter should be based on the dimensions of the entire planet, specifically, one ten-millionth of the distance from the Equator to the North Pole.

Getting Your Bearings by Keeping Time

To see why the definition of the meter came down to a choice between the swing of a pendulum and the size of the Earth takes us back to the Age of Exploration, when precision about time and space first became important in Europe. A scientific understanding of pendulum clocks and the dimensions of the planet had helped France and other countries become naval powers, since it allowed them to solve the age-old nautical problem of finding one's position at sea.

Nowadays, with GPS satellites and inertial guidance systems all around us, we tend to forget just how difficult it was for explorers like Columbus and Cartier to find their coordinates. Figuring out your *latitude*, or angular coordinate north or south of the Equator, was a fairly straightforward calculation, based on using a quadrant or sextant to record the angle of the Sun or other stars above the horizon. For example, at night in the northern hemisphere, the angle of Polaris (the North Star) above the horizon, in degrees, is almost exactly the latitude of the observer, and an astronomical table could be consulted to improve accuracy.

By contrast, figuring out your *longitude*, or east-west position, had long been a major puzzle. The ability to compute longitude was a critical national resource as European powers vied for naval supremacy and staked colonial claims across entire oceans. Two scientific academies had actually been established to find a solution to the longitude problem: the British Royal Society (1662) and the French Royal Academy of Sciences (1666). The French group had scored a "coup" by quickly hiring the Dutch physicist Christiaan Huygens, who had patented the world's first practical pendulum clock ten years earlier.

Every scientist knew that an accurate clock provided a solution to the longitude problem. Here's how: Let's say that a ship sailed west from the port of Le Havre, France with a clock set to local time. Suppose that at sea some days later, at one hour after sunrise, the clock indicated that back in Le Havre it was already 4 1/2 hours after sunrise. Then the ship would be west of port by $4.5 - 1 = 3.5$ hours (or what we might today call 3.5 "time zones"), which translates to $3.5 / 24 = 14.58\%$ of the way around the Earth. To convert this into a distance west of Le Havre, start with the total distance around the Equator, which was known to be something in the vicinity of 20,500,000 *toises* (the *toise*, defined as six "royal feet," was one of the units of length in pre-metric France). Since the distance around the Earth decreases as you move above or below the Equator (see Figure 1), the 49 1/2 degree latitude of Le Havre would have to be taken into account, leading to the conclusion that the ship was west of that port by

$$.1458 \times 20{,}500{,}000 \times \cos(49.5°) \approx 1{,}940{,}000 \text{ toises.}$$

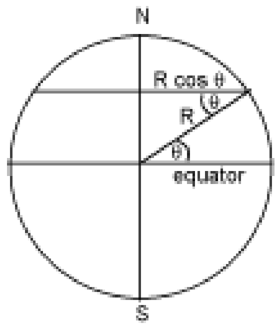

Figure 1. A cross-section shows why the distance involved in traveling around the Earth at a given latitude is proportional to the cosine of that latitude.

This strategy for determining longitude was first proposed by the Dutch mathematician Regnier Gemma Frisius in 1530. (December 8, 2008 marks the 500th anniversary of his birth.) In English, it is known as the *horological* method, horology (the word is related to "hour") being the practice of time-keeping. Notice, from the above example, that the two most difficult-to-acquire bits of information needed in the horological approach are (1) the global circumference and (2) the local time back in home port.

How a Pendulum Clock Works

It was the pendulum that first made the horological solution to the longitude problem seem potentially workable, because pendulum clocks can keep fairly accurate time through day and night, for weeks on end.

In the mathematically simplest kind of pendulum, a small object called a *bob* is attached to a long cord (see Figure 2). The bob has negligible size compared to the cord, and the cord has negligible mass compared to the bob. When given a push, the bob swings back and forth and, if the friction or resistance are negligible, no energy is lost, so the amplitude of the oscillation remains constant. Early in the 1600s, the Italian scientist Galileo had noticed that in such a *simple pendulum*, the time needed per swing is affected quite a bit by the length of the cord, but not at all by the weight of the bob, and barely at all by the amplitude of the swing.

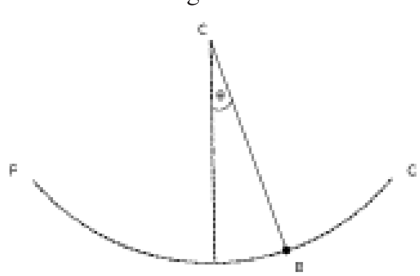

Figure 2. In a simple pendulum, the bob B traverses a circular arc PQ. If the angular amplitude of the oscillation is kept small, then its time period is nearly independent of that amplitude.

The reason for this became clear not long after the discovery of calculus. Applying Newton's laws of motion to analyze the forces acting on a simple pendulum led to a differential equation,

$$mL^2\left(\frac{d\theta}{dt}\right)\left(\frac{d^2\theta}{dt^2}\right)+mgL(\sin\theta)\left(\frac{d\theta}{dt}\right)=0$$

where θ is the angular displacement of the pendulum from its vertical rest position, t is elapsed time, m is the mass of the bob, L the length of the cord, and g the acceleration due to gravity, about 32.088 feet per second per second for any object near the Earth's surface. Upon dividing, one gets

$$\frac{d^2\theta}{dt^2}+\frac{g}{L}\sin\theta=0.$$

Because of the presence of the sine term, this is a difficult *non-linear* differential equation. But if the angle of swing, θ, remains relatively small, then $\sin\theta\approx\theta$ (in radian measure) and the equation becomes linear,

$$\frac{d^2\theta}{dt^2}+\frac{g}{L}\theta=0,$$

with general solution (check it):

$$\theta=c_1\cos\left(\sqrt{\frac{g}{L}}t\right)+c_2\sin\left(\sqrt{\frac{g}{L}}t\right).$$

Under these simplifying assumptions, then, the oscillation of the pendulum was found to have a time period

$$T=2\pi\sqrt{\frac{L}{g}}, \tag{1}$$

which varies with the length L of the cord and the acceleration g caused by gravity, but not with the mass or amplitude of the bob. Try substituting a length of three feet and see what the predicted time period is. Then, try swinging an object on a thread or string having a length of your choice, and see how well the above formula models the motion. If you tried this experiment on the surface of the moon, where the gravitational acceleration is weaker, the cord length L would need to be made proportionally smaller for the pendulum to beat at the same frequency as on Earth. (The angular amplitude, by the way, is $\sqrt{c_1^2+c_2^2}$, and is determined purely by the position and velocity that you give the bob when you set it swinging.)

Huygens improved on the simple pendulum by inventing the *cycloidal pendulum* in 1658, published later in his treatise *Horologium oscillatorium*. Brilliantly, he figured out that if the bob traverses not a circular arc but a cycloidal one (see Figure 3), then its period T is given exactly by the above formula no matter how large the amplitude of motion. (A *cycloid* is the

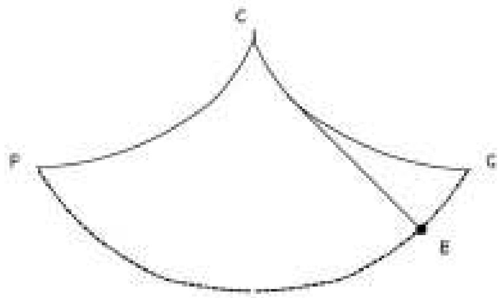

Figure 3. In a cycloidal pendulum, the bob *B* traverses a cycloidal arc *PQ*, congruent to that used to make the jaws against which the cord swings. The time period of each swing is completely independent of the amplitude.

arching periodic curve traced out by a point on a circle as the circle rolls along a straight line; it is described by the parametric equations $x = a(t - \sin t)$, $y = a(1 - \cos t)$, where a is the radius of the circle). The bob's motion is then truly *isochronous*, meaning that it takes the same time to swing regardless of its amplitude; for large amplitudes, the bob travels farther, but faster on average to make up the time.

The pendulum clock that Huygens patented was based on this cycloidal design. The cord swings against a pair of wooden planks, or "jaws," each shaped to form part of an inverted cycloidal arch. Surprisingly, as Huygens discovered, the path traced by the bob itself is an exact copy of the same cycloid used to shape the jaws. In his patent, each "beat" of the pendulum advanced a system of gears, in effect counting the seconds ticked in order to keep track of elapsed time. In practice, the design was imperfect because of the considerable friction between the cord and the jaws. But more importantly, Huygens's work on this and related problems helped lay the foundation for what came to be called the *calculus of variations*, a branch of analysis that is especially rich with applications.

Equation (1) above implies that to design a pendulum clock that beats at a certain frequency, say once or twice per second, the inventor needed to be precise about the length of the cord, but didn't need to worry about the weight of the bob or the amplitude of the swing. A frequency of one beat per second required a cord length of about 1/2 *toise* (3 royal feet), while two beats per second (i.e., one complete swing back and forth) called for a cord that was one-quarter as long. These were the most common beat frequencies used for land and marine clocks at the time.

Swing Locally, Tract Globally

The same French Royal Academy of Sciences that hired Huygens to help solve the longitude problem found itself, more than a century later, suddenly charged with inventing a revolutionary new system of units that would be "universal," in the sense of not favoring particular people or nations over others. A Commission of Weights and Measures was established by the Academy in 1790, chaired by military engineer/mathematician Jean-Charles de Borda. The other members were the renowned scientists Laplace, Legendre, Lavoisier, and Condorcet.

The French revolutionaries thought of their ideals of "liberty, equality, and fraternity" as reflecting a modern scientific outlook, and in turn these scientists charged with inventing the metric system were determined that their work would uphold those ideals. Accordingly, they searched for units of measurement that were dictated by universal principles of nature, instead of by humans with their local preferences, prejudices, and historical accidents.

The Commission considered two main strategies for setting the standard length of a meter. I like to call these the proposals to "swing locally" or "tract globally," because they were based, respectively, on local knowledge about a pendulum's motion and on global knowledge about the Earth's circumference.

The "swing locally" proposal seemed the more promising at first, because it was fairly simple. A meter would be defined as *the cord length needed to give a pendulum a frequency of one swing per second*, or one complete cycle every two seconds. This length L was already well known from the experience with pendulum clocks, as summarized above. On the Commission's recommendation, the French National Assembly adopted this proposal, introduced by Talleyrand on May 8, 1790.

But there were problems in defining the meter this way. First, it would mean that the meter's definition was based on that of a time unit (the second), making the meter a less than fundamental unit. The second itself was criticized as an arbitrary unit. (Four years later, the French government would decree a decimal system of time, but it was widely ignored and quickly abandoned.)

Even worse is the fact that the period of a pendulum varies from place to place on the Earth's surface! To see why, recall what Newton had shown: the Earth's gravitational force—including your own weight—increases as you get closer to the center of the Earth. A little calculus applied to equation (1) shows that if the gravitational acceleration increases by a certain percentage, then the pendulum's period decreases by half that percentage:

$$\frac{dT}{T} = -\frac{1}{2}\frac{dg}{g} \qquad (2)$$

In fact, if you combine equation (1) directly with Newton's Law of Gravity, you find that the time period T of a pendulum is directly proportional to its distance from the center of the Earth. This means that a pendulum swings a bit more slowly on a mountaintop than it does along a nearby coastline.

More significant is the effect of the "equatorial bulge" of the planet. Newton and Huygens had predicted that the Earth is a slightly squashed or *oblate* spheroid due to the centrifugal force of its spinning. If the planet's surface is an elliptical spheroid of eccentricity 1/230, as Newton had estimated, then the length of a pendulum would need to be varied significantly in order for it to have a 1-second swing at all localities. To test this theory, the French Royal Academy of Sciences had sent surveying expeditions to Peru and Lapland in 1735-44. Their land-survey data confirmed that the length of one degree along a meridian of longitude is greater near the Equator than near the North Pole. They were able to corroborate this finding by measuring how much more slowly a pendulum oscillated in the first location than in the second.

Clairaut, a mathematician who participated in the Lapland expedition headed by Maupertuis, published a groundbreaking study *Théorie de la Figure de la Terre* (1743) that gave a theoretical framework for the data on oblateness. He modeled the Earth as an incompressible and noncirculating fluid, rotating and subject to gravitational and centrifugal forces. In the simplified case of a two-dimensional field with components P and Q, Clairaut found that incompressibility implies that a quantity (now called "divergence") vanishes,

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}=0,$$

while noncirculation implies that a quantity (now called "curl") vanishes,

$$\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}=0.$$

He then found that oblate ellipsoids were indeed equilibrium solutions satisfying the above constraints. This work by Clairaut would be fundamental to the field of *hydrodynamics*, the mathematical study of fluid flow. It also contributed to the concept of *analytic function*, the starting point in the field of complex analysis developed in the 1800s by Cauchy and Riemann.

The findings by Clairaut, Maupertuis and others proved that defining the length of a meter by the pendulum formula (1) would mean privileging one location, above all others on the Earth, at which to base the definition. For this reason, the pendulum proposal was abandoned after less than a year. The strategy to "swing locally" turned out to be inherently too local to meet the globalist ideals of the Revolution.

Measuring the Whole Planet

On March 26, 1791 the French National Assembly accepted the rival proposal: to base the new unit on the size of the Earth itself. The meter would be defined as *one ten-millionth of the distance from the Equator to the North Pole*. Since it was the girth of the planet itself that would be measured, this proposal to "tract globally" promised to transcend local and national differences.

But this new plan was plagued by issues of its own. First, the size of the Earth was still unknown with any real precision. It is true that estimates had improved during the previous decades of marine navigation, so that it was already known that a meter, defined in this way, would end up being slightly longer than in the pendulum-based way. But to make the meter a workably precise standard of length under this proposal, the circumference of the planet would need to be measured much more accurately.

Second, it was clear that meridians of longitude varied in length because of topographic features like continents, oceans, mountains, and valleys. French officials had no choice but to select and measure a *particular* meridian, and (surprise!) they selected the one passing through their home turf, Paris.

The astronomers Méchain and Delambre were appointed to head up teams carrying out a vast land survey to tract the length of this meridian between Barcelona, Spain and Dunkerque, at the northern tip of France. The distance between these two points would then be extrapolated, using their known latitudes, to find the size of the whole planet. It was argued that this would give an accurate measurement of the Earth because (1) the effects of the globe's oblateness were thought to be negligible in France, since it is about midway between the Equator and the North Pole; and (2) the effects of topography were thought to be about average in France, since it has both mountains and coastlines.

The French surveying project involved an elaborate series of ground measurements using sightings of light signals from hilltops, church steeples, and other high points. Distances were expressed in terms of the standard *toise*, an old iron bar kept near Paris. The key mathematical method employed was *triangulation,* a technique in plane trigonometry used to determine the position of one point from those of two known points (see Figure 4). Frisius, mentioned above for his horological solution of the longitude problem, was also the inventor of triangulation. His method requires, in addition to the positions of the two known points, only the two adjacent angles of the triangle they form with the third point. The French surveying teams were able to measure these angles with great accuracy thanks to the use of a *repeating circle*, an instrument with a pair of rotating telescopes that had been cleverly designed a decade earlier by Borda, the Commission Chairman mentioned above.

The French meridian measurement entailed hundreds of observations in a vast mesh of triangles. Because the location of each new survey station was calculated relative to previous ones, in an iterative fashion, the team knew that imprecisions would tend to magnify during the surveying process. Thus,

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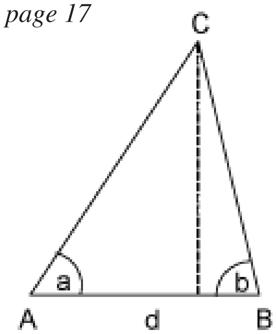

Figure 4. In surveying by triangulation, the known positions of two points *A* and *B* can be used to determine the relative position of any nearby point *C*. The calculation involves the distance *d* between the two known points, and the angular bearings *a* and *b* to the third point. First, the angle at *C* is found by subtracting from π radians the sum of the two angles *a* and *b*. The lengths *AC* and *BC* can then be computed by the Law of Sines. Next, the altitude of the triangle is found by multiplying *AC* by sin(*a*) or *BC* by sin(*b*); this gives the offset of *C* in the direction orthogonal to the line *AB*. Multiplying instead by the cosines of those angles provides offsets in the other direction.

results had to be repeatedly cross-checked and adjusted. Such long-range Earth surveying, or *geodesy*, emerged as one of the most fertile areas for the development and use of applied mathematics. To control measurement errors, you try to find approximate solutions of large over-determined systems of equations, and to apply statistical methods such as least-squares regression.

Many other problems plagued the French earth-measurement teams, some of them technical and others caused by the volatile political and military situation. What was originally seen as a one-year surveying project ended up taking seven years. Finally, in 1798, the metric standard was set and a conference was held to promote its use internationally.

The comprehensive findings of the meridian-survey effort were later published in three volumes (1806-10), which not only presented the triangulation data but included a whole history of geodesy and a full discussion of the oblateness of the Earth. Napoleon Bonaparte, the revolutionary leader and general who had crowned himself emperor, was presented with the first volume by Delambre.

Napoleon, a former artillery lieutenant, was highly trained in mathematics and science. The emperor looked over the impressive volume from Delambre and reportedly remarked, "Conquests will come and go, but this work will endure." And it has.

For Further Reading

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John L. Greenberg, *The Problem of the Earth's Shape from Newton to Clairaut: The Rise of Mathematical Science in Eighteenth-Century Paris and the Fall of "Normal" Science* (Cambridge University Press, 1995).

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