The Instability of Democratic Decisions

If issues are very simple, democratic decisions can be stable. But even a small amount of complexity destroys that stability in the most drastic possible way, and gives ultimate power to anyone who can control the voting agenda.

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In any society, people with different preferences have to decide on common courses of action. In a democratic society, we often do this by voting, usually by majority rule. When there are many possible courses of action, a decision may require a sequence of votes—a legislature might vote on amendments to amendments, then on amendments, then on a final bill. When this happens, it's important to understand how *stable* the result is. We would like to think that the result of democratic decision-making reflects "the will of the people." If the decision were made again tomorrow by the same people with the same preferences, but perhaps by a different sequence of votes, the decision should be the same, or at least approximately the same.

One way that political scientists have approached this question is by building and analyzing mathematical models. I'd like to show you how one class of models—geometric or "spatial" models—deals with the question of the stability of democratic decisions. I think you'll find the insights they offer surprising, and perhaps disturbing.

One-dimensional spatial models

The oldest spatial model in politics dates back to the time of the French Revolution, when radicals sat on the left side of the Assembly and royalists sat on the right side. Since then, common political discourse has often represented preferences by placing voters along a line, with "liberal" voters to the left, "conservative" voters to the right, and "centrist" voters in between. For example, Figure 1 shows a one-dimensional spatial model of the U.S. Supreme Court constructed by the *Washington Post* by considering how justices voted in twelve key decisions in 1998. Notice that Justice Kennedy occupies an important position, in that he is the middle, or *median*, voter. He has the swing vote between liberal Justices Souter, Breyer, Ginsburg and Stevens, and conservative Justices O'Connor, Rehnquist, Thomas and Scalia. We might expect many decisions to be made in accordance with Kennedy's preferences.

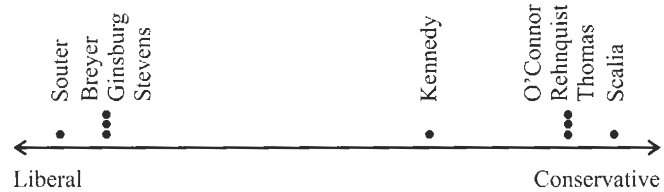

FIGURE 1. A one-dimensional spatial model of the U.S. Supreme Court in 1998.

When voters vote on alternative courses of action, those alternatives may be able to be positioned along the same continuum as the voters. For instance, voters might choose among candidates whose platforms might be liberal or conservative, or legislators might vote on bills which have liberal or conservative content. In the 1950's, Anthony Downs proposed that we analyze voting by assuming that voters and alternatives can be represented as points on a line, and that in choosing between alternatives, each voter will vote for the alternative which is closest to him or her. This is the classical "Downsian" spatial model of voting.

Let's see what the Downsian model says about the stability of voting. For simplicity, we'll always assume there are an odd number of voters, so we won't need to worry about ties. Suppose that the voters in Figure 2 choose between alternatives x and y. Notice that which alternative will win is entirely determined by whether the median voter is closer to x or y (so in Figure 2, y will win). In other words, in any pairwise majority vote, *the alternative closer to the median voter will win.* If a series of alternatives are considered, the one closest to the median voter will win, and this will be true regardless of the order in which the alternatives are voted on. The one alterna-

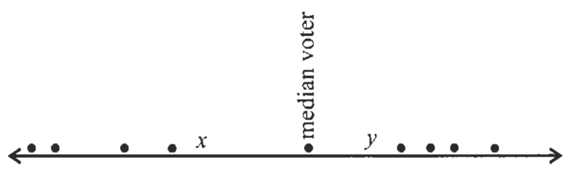

FIGURE 2. The one-dimensional Downsian voting model.

tive which will beat any other alternative is the exact point occupied by the median voter. This result is known in the political science literature as the *Median Voter Theorem*.

One way to think of this result is to imagine x and y as the positions of two candidates who can change their positions by the statements they make during the campaign. If y starts nearer to the median voter, the first candidate has incentive to move x to the right to get closer to the median voter. Then the second candidate might move y to the left to recapture the lead. We should see the positions of the two candidates converging towards the center, until there is very little to distinguish between them. (Of course, there are some practical political problems. For instance, if a candidate tries to move too far too fast, voters may question his sincerity. Or if the two candidates become indistinguishable, there may be incentive for a third candidate to enter the race taking an extreme position on the left or right.)

Although we might not like having to choose between two almost indistiguishable candidates, in many ways the Median Voter Theorem is reassuring about the nature of democratic decisions. It says they will be stable and they will be centrist—maybe not pleasing everyone but not making anyone too unhappy.

Two-dimensional spatial models

Unfortunately, we know that although democratic decisions are sometimes centrist, they are not always so. It is also true that both politicians and voters know that the liberal-conservative continuum is too simplistic to capture preferences over a broad range of issues. A politician, for example, might describe herself as "liberal on social issues, but conservative on economic issues." Mathematically, the natural way to model such preferences would be to represent voters as points not along a line, but in a plane (or a space of three dimensions or n dimensions, but for this article we'll stick to two dimensions). Alternatives—bills or candidates—are representated by points in the same plane, and we retain the Downsian assumption that in a decision between two alternatives, voters will vote for the closer one.

Surprisingly, the simple change from one dimension to two—allowing voter preferences to be slightly more complex—has profound consequences for the stability of democratic decisions. Consider, for example, Figure 3, in which there are just three voters A, B, and C who must decide on a

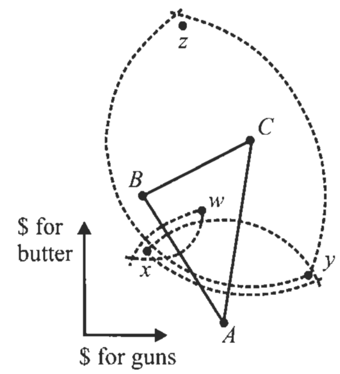

FIGURE 3. Spatial voting in two dimensions.

budget which will spend a certain amount of money for social purposes and a certain amount for military protection. If these voters are asked to decide between bills w and x, voter C will certainly vote for w, but x is closer than w to both A and B. (To make this clearer in the figure, I have shown arcs of circles centered at A and B, passing through w. Notice that x is inside both of these arcs.) Hence A and B will vote for x, which will beat w by two votes to one.

So far so good. But now suppose a new budget y is proposed, and our voters must choose between x and y. With the help of the appropriate circular arcs, you should check that A and C will vote for y, so y beats x by two to one. Finally, if z is paired against y, z will get the votes of B and C and beat y by two to one. Our small democratic society started with w and by a series of decisive majority votes, ended up choosing z. This is a strange outcome, because you can see that the voters would unanimously prefer w to z. My colleagues in the Beloit College Academic Senate find this phenomenon familiar. If you have ever been part of a legislative body, has anything like this happened to you?

McKelvey's Theorem

Let's think more carefully about the example in Figure 3. First of all, since each alternative in the cycle w, x, y, z, w would beat the previous alternative, none of those alternatives is stable. In fact, by starting the cycle in different places, we could make any one of the four alternatives the ultimate winner. The decision of this society seems to have little to do with the voters' preferences, and everything to do with the order in which alternatives are presented. This phenomenon is known in political science as the *agenda effect*. The practical consequence is that "s/he who controls the agenda, controls the outcome." In the U.S. House of Representatives, the most prized committee assignment is to the Rules Committee, which determines the voting agenda.

In Figure 3, a clever agenda controller could lead our voters from w to z. It is natural to ask where else they could be led. Political scientist Robert McKelvey published the startling answer in 1976: they can be led, by a finite sequence of majority votes, to *any point in the plane*! In two dimensions, the agenda effect is absolute.

To prove McKelvey's Theorem for the situation in Figure 3, I'll start by telling you that I didn't choose points x, y, and z at random. The circles through w centered at A and B meet at a second point, outside triangle ABC, which is the reflection of w in the line AB. To get x, I took that reflection and moved it slightly in toward the line AB. Similarly, I got y by reflecting x in the line AC and moving it in slightly. (In politics, this maneuver goes by the name of "splitting the winning coalition.") Finally, z is the reflection of y in the line BC, again moved in slightly. Of course, there is no reason we need to stop here: we could iterate the procedure of reflecting in lines AB, AC, and BC (always remembering to move in slightly) as many times as we wish. To see what happens if we do that, we need a lovely theorem from transformational geometry: *The product of reflections in three sides of a triangle is a glide reflection.* Figure 4 illustrates this result. The glide reflection which is the product of the three reflections first translates the plane along vector **v** and then reflects it across **v**. The effect of iterating the sequence of reflections, then, is to iterate a glide reflection.

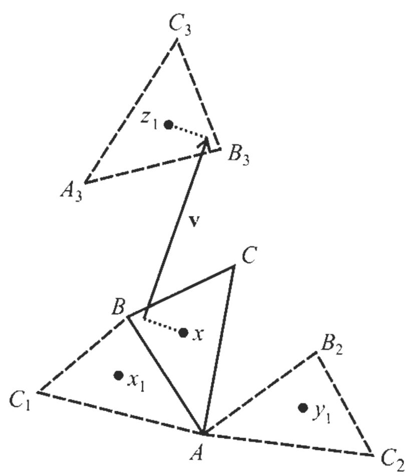

Figure 4. The product of reflections in three sides of a triangle is a glide reflection.

The strategy of our agenda controller is now clear. To lead the voters from w to any chosen point q, first iterate the glide reflection n times to get to a point z_n which is farther from all of the voters than q is. Then propose q as a final option. The voters should adopt q gratefully and unanimously.

The Plott conditions for stability

For three voters at the vertices of a triangle, there is no stability. Every alternative can be beaten by other alternatives, and in fact every alternative can be reached from every other alternative by a finite sequence of majority votes. (See how the argument above shows that?) But clearly this is a very special, and simple, configuration. What about other configurations? It

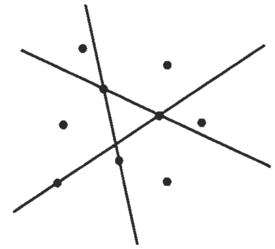

Figure 5. An unstable voting configuration of nine voters.

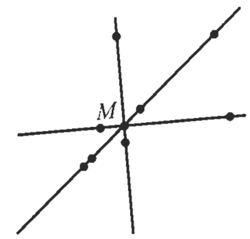

Figure 6. A stable voting configuration of nine voters.

turns out that the conclusion of McKelvey's Theorem holds "almost always." Given any configuration of an odd number of voters in the plane, define a *median line* to be a line such that each of the two regions into which it divides the plane contains fewer than half the number of voters. Figures 5 and 6 show some median lines. Notice that any median line must pass through the position of at least one voter.

The key observation is that given any alternative x, reflecting x across any median line and moving in slightly toward the median line gives a new point y which beats x under majority rule. (Can you give a quick proof of this?) Hence if there are three median lines which do not all pass through the same point, i.e., form a triangle as in Figure 5, we can use exactly the argument in the previous section to prove complete instability. Thus the only case in which instability doesn't hold is if *all median lines pass through the same point*, as in Figure 6. In this case, the common point must be the location of some voter—the *median voter M*—and all lines through M must be median lines. If M is the location of a single voter and we imagine a median line rotating around M with one half-plane picking up voters as the other loses voters, we see that whenever a median line contains k points on one side of M, it must contain exactly k points on the other side of M, i.e., the configuration must have the kind of "symmetry" of Figure 6. Configurations of this kind were first identified by Charles Plott in 1967. (If more than one voter is located at M, other configurations are possible.)

A Plott configuration is stable. You can check that, just as in one dimension, in any vote between two alternatives, the one which is closer to the median voter M will win, and an alternative positioned at M will beat any other alternative. In fact, we can think of the one-dimensional situation as a Plott configuration in which all of the alternatives happen to be positioned on one line.

Finally, notice that configurations with more than one median voter or the symmetry of a Plott configuration are extremely rare, in the sense that if we throw an odd number of points randomly on the plane, the probability that such a configuration will result is zero. Hence with probability one, a two-dimensional configuration of an odd number of voters will have complete instability.

Continued on p. 28.

acute triangle dissections in [3], problem 32. Question 10 was first proposed by H. Dudeney in 1908.

I first learned of the notion of a metric space being "triangle complete" from Tom Sibley's fascinating article [5]. In it he shows that $\mathbb{R}^3$ is not "pyramid complete" when equipped with the Euclidean metric, but is when given with the taxicab metric.

The result described in question 12 is known as Napoleon's Theorem and the generalization hinted at after its solution is due to J. Douglas and B. H. Neurmann. See G. Chang and T. Sederberg's wonderful text [1], chapter 16, for an enlightening discussion of this beautiful result.

Question 7 is a variation of Sperner's famous lemma. All one needs to ensure the existence of a 1-2-3 triangle is a labelling scheme that produces an odd number of exterior 1-2 (or 2-3 or 3-1) edges. To see how this lemma leads to the Brouwer fixed point theorem have a look at Mark Kac and Stanislaw Ulam's incredible book [4].

[1] Gengzhe Chang and Thomas Sederberg, *Over and Over Again*. The Mathematical Association of America. 1997.

[2] Martin Gardner, *Penrose Tiles to Trapdoor Ciphers*. W. H. Freeman and Company, New York, 1989.

[3] Martin Gardner, *My Best Mathematical and Logic Puzzles*. Dover Publications, New York, 1994.

[4] Mark Kac and Stanislaw Ulam, *Mathematics and Logic*. Dover Publications, New York, 1992.

[5] Tom Sibley, The possibility of impossible pyramids, *Mathematics Magazine* 73 No. 3 (2000), 185–193.

Continued from p. 14.

Morals of this story

The results of our foray into spatial models of voting seem discouraging, perhaps even dismaying, for those who believe in majority rule as a method of democratic decision-making. If issues are very simple, in that it makes sense to represent voters' positions along a line, decisions can be stable. But even a small amount of complexity destroys that stability in the most drastic possible way, and gives ultimate power to anyone who can control the voting agenda.

What should we conclude from this? One possible response—and one which has sometimes been made by non-mathematical political scientists—is that this is "only a model," with no relation to complex reality. The problem with this dismissal is that other political scientists recognize real political behavior in what the model portrays: that different procedural rules can produce different outcomes, that minorities can propose amendments which split majorities, that control of the voting agenda can be enormously important.

An extreme response on the other side is that the model has uncovered a fundamental flaw in democratic society. It shows that in any society of even moderate complexity, making decisions by majority rule gives power not to the people, but to professionals who know how to manipulate political procedures to their own advantage.

A more moderate response, and one which has been influential in political science for the past twenty years, comes from focusing on the aspect of the model which produces the instability: the freedom to introduce new alternatives anywhere in the plane. This line of research asks what kinds of "germaneness" rules, parliamentary procedures and voting rules might best promote stability without limiting in too serious a way the freedom of voters to propose alternatives.

For me, a main moral is that mathematical models can focus ideas, bring new insights, and suggest new lines of research in the study of important social questions. ■

For Further Reading

A good general introduction to spatial voting models can be found in J. Enelow and M. Hinich, *The Spatial Theory of Voting* and *Advances in the Spatial Theory of Voting*, Cambridge University Press (1984) and (1990). A nice proof that the product of reflections in three sides of a triangle is a glide reflection can be found in I. M. Yaglom, *Geometric Transformations I*, Mathematical Association of America, 1962. A more detailed discussion of voting instability appears in P. D. Straffin, Power and stability in politics, pp. 1128–1151 in R. Aumann and S. Hart, eds., *Handbook of Game Theory with Economic Applications*, vol. 2, Elsevier (1994).

ERRATA

In the November 2001 WordWise (Another Verse, Changed from the First) it was stated that there are 12 permutations of order six among the 720 possible permutations of six objects. Thanks to the readers who pointed out that there are, of course, 120. Our apologies to readers who were misled.

The final sentence of the article Packing Rectangles with the L and P Pentominoes in the November 2001 *Math Horizons* contained an error. That sentence should read:

> The smallest prime is the 5×10 rectangle, and there are 39 other primes, the largest being the 12×95 rectangle.