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# The Parallel Climbers Puzzle

## A Case Study in the Power of Graph Models

Two climbers start at points A and Z on the left and right sides, respectively, of the mountain range in Figure 1. We pose the following puzzle, which we call the Parallel Climbers Puzzle.

**Parallel Climbers Puzzle:** Is it possible for the left and right climbers to move from *A* and *Z*, respectively, along the range in Figure 1 to meet at *M* in a fashion so that *they are always at the same altitude every moment?*

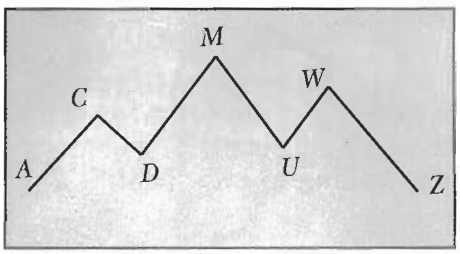


*Figure 1*

Our objective of this paper is to show that this puzzle has a solution for any mountain range. The two assumptions we make are that: i) *A* and *Z* are at the same height, and ii) there is no point lower than *A* (or *Z*) and no point higher than *M*. We shall solve this problem by means of a graph theoretic model.

Graph theory is an important field of discrete mathematics. While the term 'graph' is used most commonly in mathematics to refer to the set of points satisfying a functional relationship between two (or more) variables, the word 'graph' has another meaning as a mathematical object $G = (V,E)$, consisting of a non-empty, finite set $V$ of **vertices** and a set $E$ of **edges** joining certain pairs of vertices. Figure 2 shows a graph $G = (V, E)$ with vertex set $V = \{a, b, c, d, e\}$ and edge set $E = \{a\text{-}b, a\text{-}d, b\text{-}c, c\text{-}d, c\text{-}e\}$. (Note that an edge such as $a\text{-}b$ is normally written as the ordered pair (a,b), but this standard notation turns out to be confusing in this particular modeling problem, because we need ordered pairs for another purpose.) The mountain range depicted in Figure 1 can be viewed as a graph with vertex set $V = \{A, C, D, M, U, W, Z\}$ and edge set $E = \{A\text{-}C, C\text{-}D, D\text{-}M, M\text{-}U, U\text{-}W, W\text{-}Z\}$.

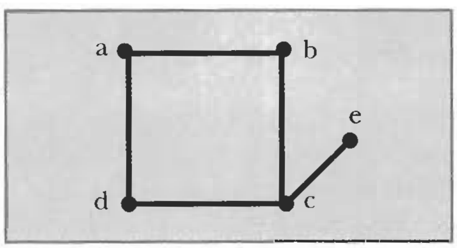


*Figure 2*

Graphs are used to analyze a wide variety of mathematical problems, most commonly in operations research and computer science. For example, graphs can represent transportation and telecommunication networks. The problem of simultaneously routing thousands of long-distance calls between various pairs of parties is a graph theory problem. The data structures of computer science that are used to organize linkages between various pieces of information are graphs.

An example of such a data structure is a search tree for looking up a word in a spell-checking dictionary. The spell checker does not start at the beginning of the dictionary and sequentially check the unknown word against all 50,000 words in the dictionary. Even for a fast computer, that would be unnecessarily time-consuming. Rather, it compares the unknown word with the middle word in the dictionary's alphabetical list of words to see if the unknown word occurs in the first half or second half of the dictionary. Depending on the outcome of the first test, the spell checker next compares the word with the middle word in the first half or second half of the dictionary, and so on. This strategy uses each comparison to divide the set of dictionary words that need to be checked in half. The data structure organizes the comparisons, telling what comparison to do next depending on the outcome of the current comparison. Figure 3 shows the beginning of a possible data structure graph for spell checking.

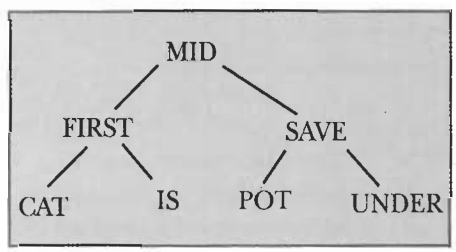


*Figure 3*

A related but simpler type of search problem involving graphs is the following: can one find a path—a sequence of connecting edges-from a specified starting vertex to a specified stopping vertex

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in a given graph. The vertices in the graph might represent all the possible positions in some puzzle and the edges legal moves between positions in the puzzle. "Solving" the puzzle reduces in the associated puzzle graph to finding a path from the starting position to the stopping, or winning, position.

The mountain climbing problem posed above is such a puzzle with a peak or valley (the other point might also be a peak or valley). An edge in a range graph will join two vertices ($P_L$, $P_R$) and ($P'_L$, $P'_R$) if and only if the two people can move constantly in the same direction (both going up or both going down) from point $P_L$ to point $P'_L$ and from $P_R$ to $P'_R$, respectively.

First we redraw the mountain range in Figure 1 as shown in Figure 4 with of the great values of graph models. They can often reformulate problems in a fashion that makes the answer easy to "see." However graphs also provide a framework for proving general results. First, we shall introduce the concept of the degree of a vertex and make two simple observations about degrees. This information, when applied to range graphs, will then lead to a simple proof

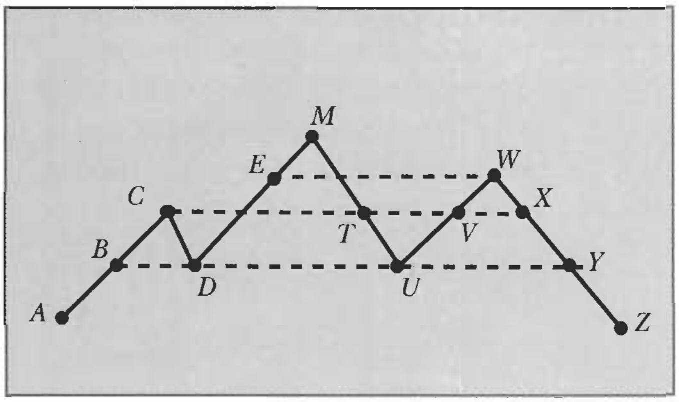


*Figure 4*

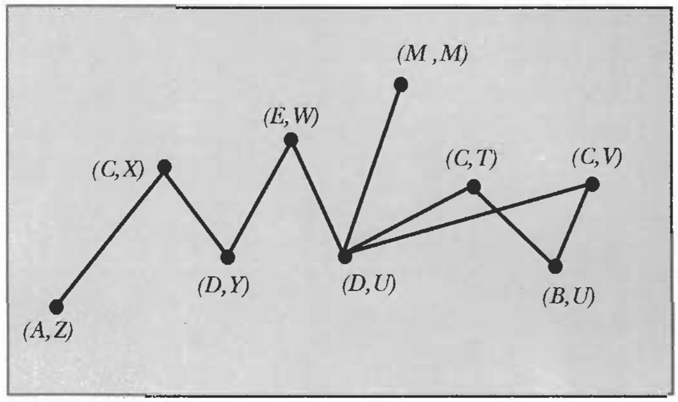


*Figure 5*

starting position—the left and right climbers starting at locations *A* and *Z*, respectively—and a stopping positions—both climbers at location *M*. The challenge in creating such a puzzle graph is to determine what should be the set of vertices representing possible positions of interest. There are an infinite number of points along the initial ascend segments the two climbers take, but a graph is defined to have a finite number of vertices.

Like the spell-checker problem, we are concerned in the Parallel Climbers problem with making a sequence of decisions. These decisions occur at places along the climb where the climbers will have choices. That is, in the mountain range in Figure 1, when one of the climbers arrives at a peak or valley (the labeled points in Figure 1), we must correctly decide what should be done next. Thus, the peaks and valleys are the locations of interest that should be used to define the vertices in our graph model.

A **range graph** is a graph whose vertices are pairs of points ($P_L$, $P_R$) at the same attitude with $P_L$ on the left side of the summit and $P_R$ on the right side, such that one of the two points is a local

points added parallel to peaks and valleys. For example, when the left climber is at *C*, the right climber could be at point *X* or point *V* or point *T*. Thus (*C,X*), (*C,V*) and (*C,T*) will all be possible vertices in the range graph, since the two climbers at some stage might come to any of these three pairs of locations.

The range graph for the mountain range in Figure 4 is shown in Figure 5. Our question is now, Is there a path in the range graph from the starting vertex (*A, Z*) to the summit vertex (*M, M*). For the graph in Figure 5, the answer is, by inspection, obviously yes.

Following the path in Figure 5 from (*A,Z*) to (*M,M*) takes us first up to (*C,X*), then the left climber moves lower but still towards the summit, while the right climb backs down, coming to (*D,Y*). Next both climbers move upwards to (*E,W*). Next the left climber backs down while the right climber moves lower but towards the summit, coming to (*D,U*). From there, the climbers can jointly ascend to the summit, (*M,M*).

The model we made in Figure 5 recasts the puzzle in a fashion that made it easy to solve by inspection. This is one

that every Parallel Climbers puzzle has a solution.

The **degree** of a vertex is the number of edges incident to that vertex.

For simplicity, we shall assume that the two ends of an edge must be distinct vertices.

*Theorem: The sum of the degrees of all vertices in a graph is equal to twice the number of edges in the graph.*

Proof: The sum of all degrees counts every occurrence of an edge being incident to a vertex. Since every edge has two endvertices, twice the number of edges also counts every occurrence of an edge being incident to a vertex. The theorem follows.

*Corollary: In any graph, there must be an even number of vertices of odd degree.*

Proof: Since the sum of all degrees is an even number—namely, twice the number of edges, by the Theorem—then the number of odd terms in this sum must be even.

Now let us examine the degrees of the vertices in a range graph. We claim that vertices (*A, Z*) and (*M, M*) in any range graph have degree 1 while every other vertex in the range graph has degree 2 or 4. (*A, Z*) has degree 1 because when both people start climb-


*Illustration by Loel Barr*

ing up the range from their respective sides, they have no choice initially but to climb upwards until one arrives at a peak. In Figure 4, the first peak encountered is *C* on the left, and so the one edge from (*A, Z*) goes to (*C, X*). A similar argument applies at (*M, M*). Next consider a vertex ($P_L$, $P_R$) where one point is a peak and the other point is neither peak nor valley, such as (*E, W*). From the peak we can go down in either direction: at *W*, we can go down toward *Z* or toward *U*. In either direction, the people go until one (or both) climbers reach a valley. At (*E, W*), the two edges go to (*D, Y*) and (*D, U*). So such a vertex has degree 2. A similar argument applies if one (but not both) points are a valley. It is left as an exercise for the reader to show that if a vertex ($P_L$, $P_R$) consists of two peaks or two valleys, such as (*D,U*), it will have degree 4. (A vertex consisting of a valley and a peak will have degree 0-why?)

Suppose there were no path from (*A, Z*) to (*M, M*) in the range graph. We use the fact that the starting vertex (*A, Z*) and the summit vertex (*M, M*) are the only vertices of odd degree. The part of the range graph consisting of (*A, Z*) and all the vertices that can be reached from (*A, Z*) would form a new graph with just one vertex of odd degree, namely, (*A,Z*). This contradicts the corollary and so any range graph must have a path from (*A, Z*) to (*M, M*).

Thus, we have proved:

*Theorem: The Parallel Climbers puzzle has a solution for any mountain range.*■