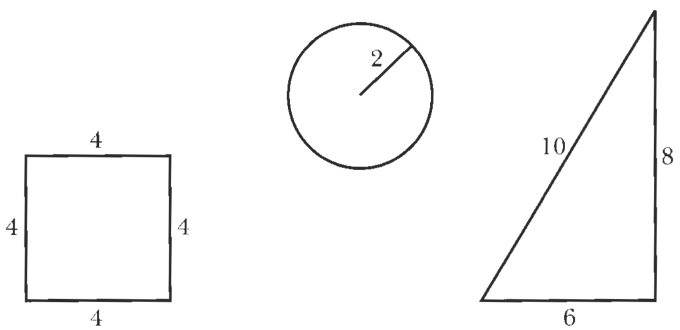
JAMES TANTON

A Dozen Areal Maneuvers

Your dozenal correspondent is at it again! This time I have put together a collection of twelve curiosities all to do with areas, and, in some cases, the perimeters that contain them. The questions about slicing pie and cake are technically ones about volumes, but we'll assume here all desserts are of uniform thickness so they may be reduced solely to analysis of area! Many of these results are classic (one even known by Archimedes) but hopefully the few extra twists I've put in shine these gems in a new and interesting light. I hope you have as much fun thinking about these as I did.

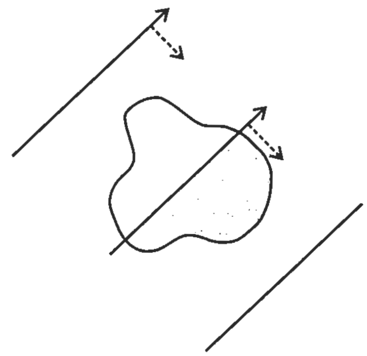
1. Plucky Perimeters

What curious property do the following figures share?



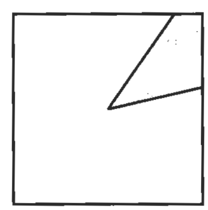
2. Irregular Pizza

Much to their dismay, Sam and Maggie receive from their local pizza parlor an irregularly shaped pizza. Both being mathematicians, they realize the Intermediate Value Theorem assures them the existence of a straight line cut that divides the pizza exactly in half: By sliding the knife across the pizza, first from a position with all the area of the pizza sitting to the right of the knife, to one with all the area sitting to the left, there must be some intermediate position where the area is split precisely in two.



Is there necessarily a straight line cut that not only divides the area of the pizza precisely in half but also the pizza crust (that is, the perimeter of the figure) in half as well?

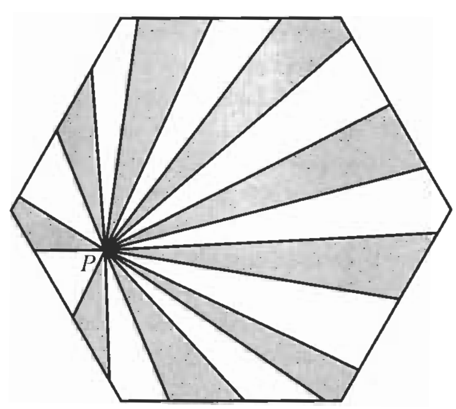
3. Square Pie



Cutting a wedge emanating from the center, Beverly wants to take precisely one-seventh of a square pie. She doesn't like crust. Where should she position her cut so as to receive the minimal length of perimeter?

4. Hexagonal Pie

Beverly cut eighteen slices into a hexagonal pie. She missed the center of the pie but managed to ensure that each wedge-shaped piece possessed the same length of perimeter.



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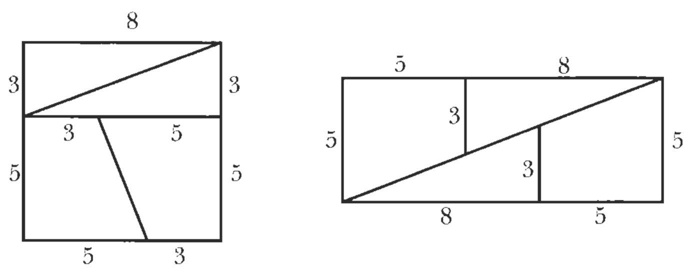
Prove that the total area of the shaded regions (every second piece) equals the total area of the unshaded pieces, and that this is always the case no matter where the "center" point P is placed.

5. Cake Sharing

Is it possible to share a cake among three people so that each person honestly believes she is receiving *more* than one-third of the cake?

6. Creating Area

Who said area is always preserved? Take an 8×8 inch square piece of paper and subdivide it as shown on the left.

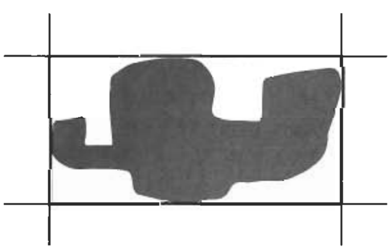


Now rearrange the pieces to form a 5×13 rectangle as shown on the right. (Try it!) This transforms 64 square inches of paper into 65 square inches.

What's going on?

7. Capturing Area

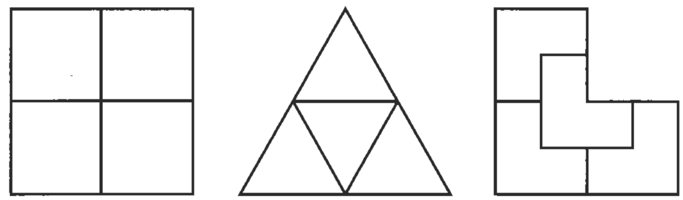
It is always possible to capture any given shape within a rectangular box. Simply slide in four straight lines, one from each direction of a compass, north, south, east and west, until they each just touch the given region.



Is it always possible to capture a region within a perfectly square box?

8. Rational Replication

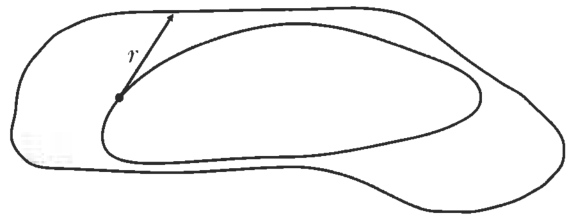
Four squares stack together to form a larger copy of themselves, as do four equilateral triangles, and four bent trominoes. The larger figures are scaled versions of the original tiles, each with rational scaling factor 2.



Is there any figure in the plane that replicates itself with fewer than four copies to produce a larger copy still with rational scale factor?

9. Bicycle Tracks

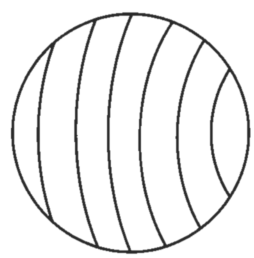
A bicycle of length r (measured as the distance between the points of contact of the two wheels with the ground) moves along a closed convex loop.



What is the area between the two tracks it leaves?

10. Spherical Bread

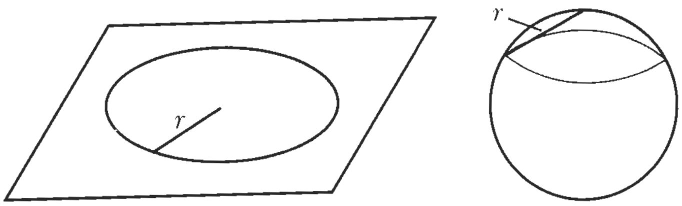
A spherical loaf of bread, n units in diameter, is sliced into n pieces of equal thickness.



Which piece has the most crust?

11. Circles on Spheres

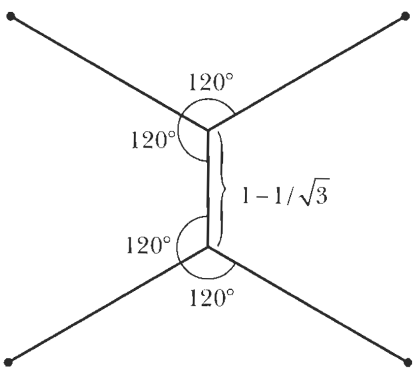
Which is greater: The area of a circle of radius r drawn on a plane, or the surface area of a circle of "radius" r drawn on



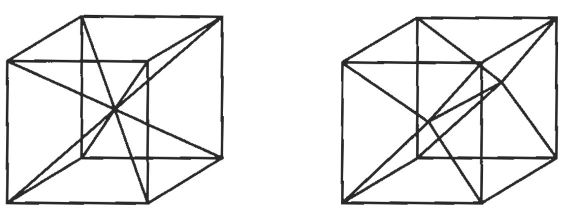
a sphere? (Here "radius" is the length of the straight line segment passing through the interior of the sphere connecting the center of the circle to its perimeter.)

12. Soap Film on Cubical Frames

A classic problem asks what system of roads connects four houses situated on the vertices of a square (one mile wide) using minimal total road length. The answer, surprisingly, is the wing-shaped design below. It uses $1 + \sqrt{3} \approx 2.732$ miles of road.



My aim here is to take this problem up a dimension. What design of surfaces, meeting somewhere in the center, connects the skeleton of a cube (namely its 12 edges and 8 vertices) with minimal total surface area?



This problem is very difficult to analyze mathematically, but with the aid of soap solution the answer can be determined experimentally. Using pliable wire make a cubical frame and dip it into soap solution. The surface tension of the film acts to minimize surface area and so careful dipping (making sure the film is attached to every edge of the cube and meets in the center) will result in the desired solution. Try it! What's the answer?

Answers, Comments, and Further Questions

1. Their perimeters equal their areas! Of course this is just an artifact of scale. With the appropriate enlargement or reduction it is theoretically possible to scale *any* figure so that its perimeter equals its area. (Challenge: Use a photocopier to produce a reduced copy of this very page with perimeter equal to area, measured in inches and square inches.)

Taking it Further Find a non-square rectangle with integer side lengths whose perimeter equals its area. Are there any more such rectangles? There is only one other right triangle with integer side lengths having this property. What is it?

Taking it Even Further Is there a rectangular box whose volume equals both its surface area and the total sum of its edge lengths?

2. Sam and Maggie's argument using the Intermediate Value Theorem shows for *any* given angle θ there is a unique directed line tilted at that angle dividing the area of the pizza precisely in half. (We regard a line at angle θ and a line at angle $\theta + 180°$ as distinct lines pointing in opposite directions.) Let's measure how successful these lines are in cutting the perimeter in half as well. For each angle θ set $f(\theta)$ to be the total length of the crust to the right of the line minus the length of the crust to its left. Our goal is to find a line with $f(\theta)=0$.

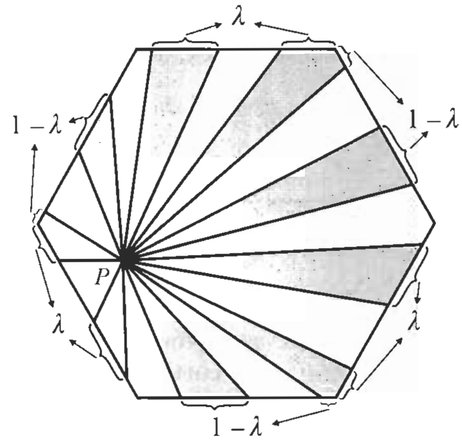
Notice that $f(0°)$, whatever its value, equals $-f(180°)$. These angles represent the same line but pointing in opposite directions. Since f varies continuously with the angle θ, the Intermediate Value Theorem tells us there must indeed be an angle θ with $f(\theta)=0$. (The continuity of f is subtle.) This does the trick.

Taking it Further Prove there is always a (very long) single straight line cut that simultaneously slices any *two* irregularly shaped pizzas in half, no matter where they are placed on the table top. Can one always simultaneously divide *three* pizzas in half in a single straight cut?

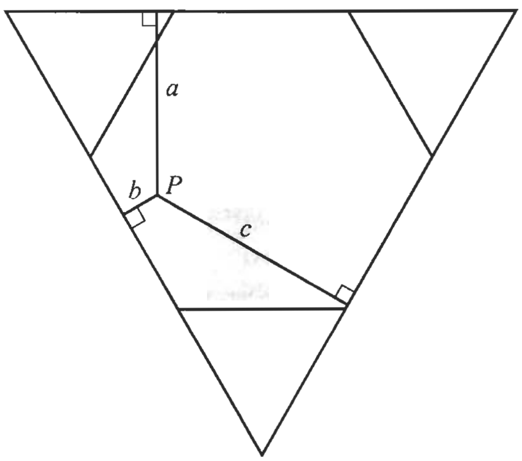
3. It does not matter where she places her cut: All wedges from a square pie possess the same portion of perimeter! Pieces of pie are either triangular or a union of two triangles. As all these triangles have the same height, the area of any wedge is directly proportional to the length of perimeter it contains. Thus one-seventh of the area always means one-seventh of the perimeter. Note there is nothing special about the fraction "one-seventh" nor the square shape. The same phenomenon occurs when taking slices from the center of any regular polygon.

4. Assume the hexagon has unit side length. The height of the hexagon (shortest diameter) is thus $\sqrt{3}$. By the regularity of the situation, if λ% of the top edge length belongs to shaded wedges, $(1-\lambda)$% of the opposite edge length belongs to them too. In fact, the portion of edge belonging to shaded regions alternates between λ% and $(1-\lambda)$% around the figure.

Changing track for the moment: Let a, b, and c be the distances to every other side of the hexagon. See the figure below. View these as line segments in the interior of a large equilateral triangle. A result from geometry says that the sum of these lengths equals the height of the triangle. Thus $a + b + c = 3\sqrt{3}/2$, no matter where P happens to lie. (To establish this, first consider the case where P lies on the base



edge of the triangle. The result is clear from drawing a reflected image of the triangle across this edge. To establish the general case, raise the base of the triangle so that P lies on the base edge of a sub-equilateral triangle.)



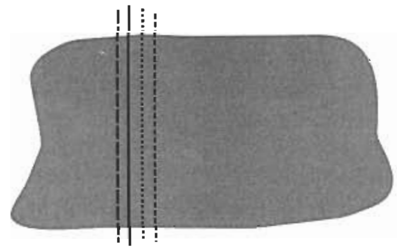
Regarding the shaded pieces encompassing a corner of the hexagon as a union of two triangles, we have that the total area of the shaded triangles touching the top and bottom edges is $\frac{1}{2}\lambda a + \frac{1}{2}(1-\lambda)(\sqrt{3} - a)$. Similarly for the remaining two pairs of edges. Summing and simplifying thus gives the total area of the shaded regions to be:

$$\frac{1}{2}\lambda(a+b+c)+\frac{1}{2}(1-\lambda)\left(3\sqrt{3}-a-b-c\right)$$
$$=\frac{1}{2}\lambda\frac{3\sqrt{3}}{2}+\frac{1}{2}(1-\lambda)\frac{3\sqrt{3}}{2}$$
$$=\frac{1}{2}\cdot\frac{3\sqrt{3}}{2}$$

which is precisely half the area of the hexagon.

Taking it Further Every third region of the eighteen slices is selected. Prove these sum to one-third of the area of the hexagon. Suppose instead Beverly makes just 12 cuts. Prove every second piece, and then every fourth piece, account for precisely one-half and one-quarter respectively of the pie. Can you extend these results to other numbers of cuts? To other regular polygons?

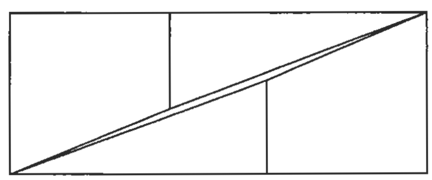
5. If everyone possesses a different estimation of "one-third," this seemingly impossible task is then indeed possible! Here's one scheme: First have each person score a straight line across the cake, in parallel, at a position she honestly believes cuts off one-third of the cake from the left. Then make a cut anywhere between the two leftmost lines and hand that piece to the person who marked the line closest to the end. This person is receiving more than one-third of the cake in their estimation, and the two remaining folks believe more than two-thirds remains.



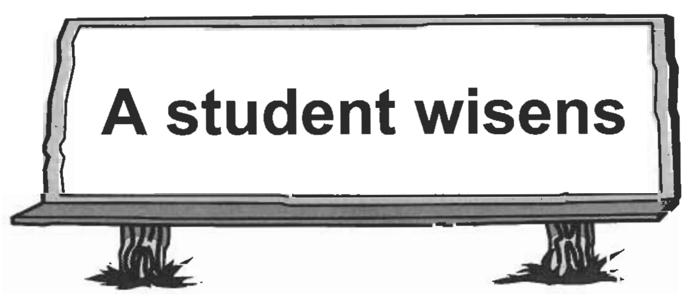
Have these two folks then each mark a line dividing the remaining portion precisely in half in their estimation. Cutting the cake between these two lines lands each person with more than half of more than two-thirds of the cake! This does the trick.

Taking it Further Devise a cake cutting scheme between three people that not only assures everyone at least one-third of the cake in his estimation, but also the biggest (or tied for biggest) piece ever cut!

6. If you look carefully at the rectangular arrangement of the 8 × 8 square you will notice that the pieces don't quite line up correctly. We usually deem such discrepancies as due to imprecise cutting, but in this case the errors are inherent to the problem. There is a gap in the middle of the rectangle that accounts for the missing unit area.

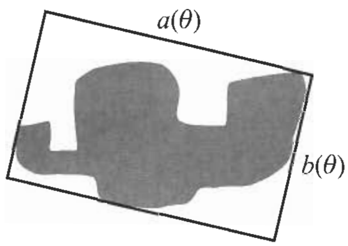


Taking it Further Take any three consecutive integers F_{n-1}, F_n, F_{n+1} from the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13,



21, 34, 55,... (defined recursively by $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$). Show how to transform an $F_n \times F_n$ unit square into an $F_{n+1} \times F_{n-1}$ rectangle. Have you again lost track of a square inch of paper?

7. Every closed and bounded planar region can indeed be captured within a square box! For each angle θ we can certainly find a rectangular box tilted at that angle that captures the region.



Let $a(\theta)$ and $b(\theta)$ be the side lengths of that box and set $f(\theta) = a(\theta) - b(\theta)$. As $f(\theta + 90°) = -f(\theta)$, the Intermediate Value Theorem guarantees an intermediate value θ^* with $f(\theta^*) = 0$. The rectangle at this angle is a square.

Taking it Further Can every closed, bounded planar region be captured by an equilateral triangle? By a regular pentagon?

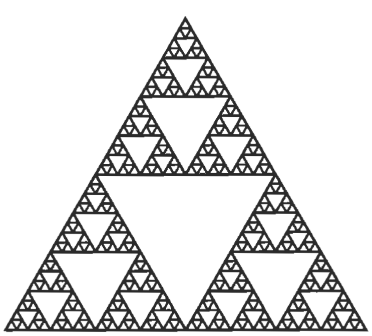
8. Suppose lengths scale by a factor k. Then area scales as k^2. Suppose there is a figure in the plane that replicates itself with just two copies. Then

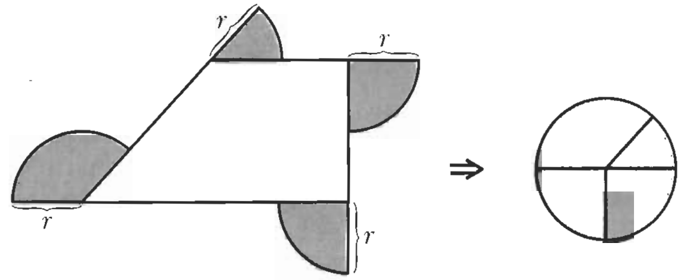
$$\begin{aligned} 2 \times \text{Area (small figure)} &= \text{Area (large figure)} \\ &= k^2 \times \text{Area(small figure)}. \end{aligned}$$

Necessarily $k = \sqrt{2}$, an irrational number. Similarly, we must have $k = \sqrt{3}$ for any three-self replicating tile. We need four (or nine, sixteen, ...) tiles to produce a rational scale factor.

Taking it Further Find examples of self-replicating figures that replicate with just two and three tiles (necessarily with irrational scale).

Note If we extend our notion of "planar figure" to include fractal figures (that is, objects whose "area" scales by a factor k^d with $d \neq 2$) then it is possible to find self-replicating objects that produce rational scaled copies of themselves with fewer than four copies. The Sierpinski triangle is an example of such an object, replicating itself with just three copies with scale factor $k = 2$. (Here $d = \ln 3/\ln 2 \approx 1.585$.)

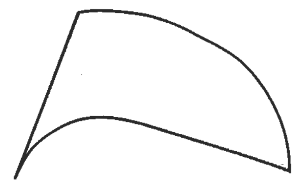




9. Note that because the back wheel is fixed in its frame, the tangent line to the inner curve (the back wheel track) always intercepts the front track a fixed distance r along the direction of motion. (See [5].)

First consider the case where the back wheel travels along the edges of a convex polygon, turning sharply at each corner (in fact, pivoting about the point of contact.) The front wheel travels in straight lines as the back wheel follows the edges, and sweeps out sectors of a circle of radius r at each corner. These sectors fit together to form a complete circle of radius r, and hence the area between the two tracks in this polygonal case is πr^2.

Any curve can be approximated by a polygonal curve. By a limit argument we thus deduce the area between bicycle tracks is always πr^2.



Taking it Further What can you say about the area between two bicycle tracks along non-convex curves? (See [4].) If a bicycle follows only a portion of a curve, turning a total angle θ in the process, what can you say about the area between the two curve segments?

10. A sphere (of radius R) is obtained by revolving the graph

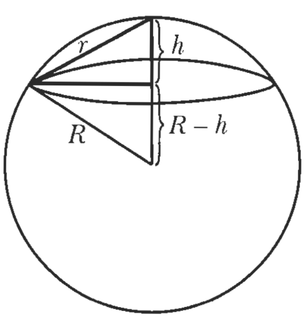
$$f(x) = \sqrt{R^2 - x^2}$$

about the x-axis. From calculus, the surface area of a segment between positions $x = a$ and $x = a + h$ is given by:

$$\begin{aligned} \int_a^{a+h} 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx &= \int_a^{a+h} 2\pi R\, dx \\ &= 2\pi Rh \end{aligned}$$

Thus all slices of thickness h have the same surface area. In terms of our spherical loaf of bread this means all slices have precisely the same area of crust! (This result was known to Archimedes.)

11. Suppose the sphere has radius R. Let h be the distance indicated. By the Pythagorean theorem (twice), $r^2 - h^2 = R^2 - (R - h)^2$. Consequently, $2Rh = r^2$. Now by question 10 the surface area of this slice of sphere of



Continued on p. 34.

Also solved by Georgi D. Gospodinov (student), Mark A. Mills, Robert Feinglass (student), Nigel Salts, Mark Shattuck (graduate student), Anna Sortland (student), Michael Woltermann, and the proposer.

Problem 130. Triangles with $\angle A = 3\angle B$

Determine an infinite set of non-similar triangles ABC of integer sides a, b, c such that $\angle A = 3\angle B$.

The solutions by Donald J. Moore, Wichita KS and the Problem Editor were essentially the same and constitute all possible solutions. Since $a = 2R\sin A$ etc., $a/b = 4\cos^2 B - 1$, $c/b = 4(\cos B)(2\cos^2 B - 1)$ so that $c = (a-b)\sqrt{1+a/b}$. Thus, $1 + a/b = (r/s)^2$ where $(r,s) = 1$. Then $a = (r^2 - s^2 b)/s^2$ and $c = (r^2 - 2s^2)rb/s^3$. To be integers, $b = ns^3$ and then $a = ns(r^2 - s^2)$, $c = nr(r^2 - 2s^2)$. In order that a, b, c be sides of a triangle, we must have $2 > r/s > \sqrt{2}$. One simple example is $a = 10$, $b = 8$, and $c = 3$. Another set of solutions is gotten by changing n and s to $-n$ and $-s$, giving $a = ns(r^2 - s^2)$, $c = nr(2s^2 - r^2)$, $b = ns^3$ where now we must have $\sqrt{2} > r/s > 1$.

Also solved by Mark Shattuck (graduate student), and Michael Woltermann.

Editorial Note. After this issue the Problem Section will be back on schedule. There was no Feb. 2000 Problem Section since I was hospitalized for 95 days. I am grateful to the guest editors, Titu Andreescu and Kiran Kedlaya, for the April section. Since in that section there were many solvers who were not acknowledged (the solutions had accumulated in my school mail box), I am acknowledging them now:

S-32. *The Problem Editor (by vectors), and the proposers.*

S-34. *Cabral Balreira, Etienne Cupuis, Daniel Hermann, Mica James, Natasha Keith, Koopa T.L. Koo, Martin Mak, Jason E. Parker, Tim Pope, Westmount College Problem Solving Group, and the proposer.*

Problem 123. *Amritpreet Singh, David Vella, Westmount College Problem Solving Group, and the proposer.*

Problem 124. *George Delgado, Jason E. Parker, Westmount College Problem Solving Group, Michael Woltermann, and the proposer.*

Problem 125. *Angelo State Problem Group, GVSU Problem Group, Micah James, Tim Pope, Ben Schmidt, Randy K. Schwartz, Skidmore College Problem Group, Anna Sortland, SUNY Fredonia Student Group, Westmount College Problem Solving Group, Michael Woltermann, and the proposer.*

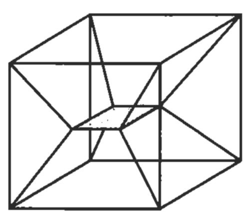
Problem 126. *Angelo State Problem Group, Michael Woltermann, and the proposer.*

Continued from p. 30

thickness h is $2\pi Rh = \pi r^2$. This is the same area as the planar circle!

Taking it Further What is the area between two bicycle tracks on a sphere?

12. In analogy to the two-dimensional problem, the soap solution forms a small square of film hovering in the center of the cube.



What happens if you gently tap this structure?

Taking it Further Notice that four edges of film meet at every interior vertex and that any two films meeting at an edge do so at an angle of 120°. In 1976 F. J. Almgren and J. E. Taylor proved that all such soap film structures behave this way. With this in mind would you care to predict what results when the frame of a tetrahedron or a triangular prism is dipped in soap solution?

Acknowledgments and Further Reading

Many of these puzzles appear in my forthcoming mathematical activities book (see [5]) along with further analysis connecting them to other branches of mathematics. Problem 4, in some sense, is a discrete version of a problem that appears in J. Konhauser, D. Velleman and S. Wagon's truly wonderful text [2] (problem 63). (How does the novel solution presented there apply to our situation?) Problem 11 is also from this text. Areas swept out by tangent line segments are examined in M. Mnatsakanian's delightful piece [4] (though bicycles are never mentioned). Combining problems 9 and 11 leads to interesting thoughts about bicycles on spheres. The sharing of cake is a classic topic in mathematics. For a very accessible account of this see [1], chapter 13. Other intriguing soap film questions and experiments can be found in F. Morgan, E. Melnick and R. Nicholson's fabulous article [3]. ■

1. COMAP Inc., *For All Practical Purposes: Introduction to Contemporary Mathematics*, 4th ed., W. H. Freeman & Co., New York, 1997.
2. J. Konhauser, D. Velleman, S. Wagon, *Which Way did the Bicycle Go? and other Intriguing Mathematical Mysteries*, Dolciani Mathematical Expositions No. 18, The Mathematical Association of America, Washington, DC, 1996.
3. F. Morgan, E. R. Melnick, R. Nicholson, "The soap-bubble-geometry contest," *The Mathematics Teacher*, **90**, No. 9 (1997), pp. 746–749.
4. M. Mnatsakanian, "Annular rings of equal areas," *Math Horizons*, November 1997, pp. 5–8.
5. J. S. Tanton, "A half-dozen mathematical activities to try with friends," *Math Horizons*, September 1999, pp. 26–31.