Curriculum Burst 7: Circle Area

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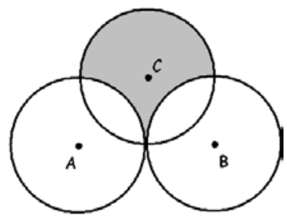
Circles A, B, and C each have radius 1. Circles A and B share one point of tangency. Circle C has a point of tangency with the midpoint of $\overline{AB}$. What is the area inside circle C but outside circle A and circle B?



SOURCE: This is question # 18 from the 2011 MAA AMC 10a Competition.

QUICK STATS:

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MAA AMC GRADE LEVEL

This question is appropriate for the 10^{th} grade level.

MATHEMATICAL TOPICS

Geometry: Areas of circles and sectors.

COMMON CORE STATE STANDARDS

FIND ARC LENGTHS AND AREAS OF SECTORS OF CIRCLES:

G-Q.5: Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGIES

ESSAY 5: **SOLVE A SMALLER VERSION OF THE SAME PROBLEM**

THE PROBLEM-SOLVING PROCESS:

As always… the most important step:

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

I have two reactions. One is slight panic as I have to get "geometry speak" back in my head ("circle A" and "$\overline{AB}$" and "point of tangency," etc.) The second is: Why? Why do I want the measure of this area?

Okay. Deep breath. I guess I care because it is an interesting puzzle. That's fine. There is nothing wrong with puzzling out something for the fun of it.

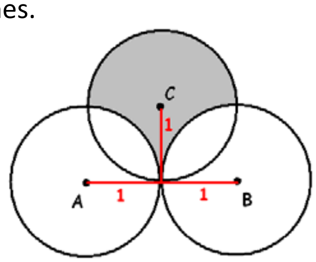
But the jargon:

"Circle A:" I guess that means a circle with center A.

"Point of tangency" means where the point at which figures touch. (Doesn't the word *tangent* come from the Latin word "to touch"?)

The notation $\overline{AB}$ must mean the line segment connecting A and B. (What else could it be in this question?)

We are told each circle has radius 1. It seems compelling to draw in some lines.

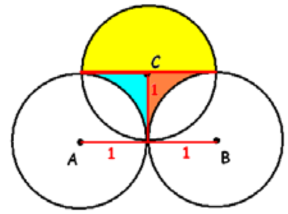


It seems that this illustrates the description: *"Circle C has a point of tangency with the midpoint of $\overline{AB}$."*
Also, aren't there results in geometry about tangent lines and radii being 90° to each other? I am guessing the radii we've drawn here are perpendicular.

Okay. Now what?

SOLVE A SMALLER VERSION

We want to find the area of the shaded portion. It seems natural to break it into smaller parts.



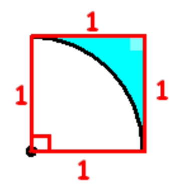
The top yellow portion is a semicircle of a circle of radius 1. Its area is easy:

$$\frac{1}{2}\pi r^2 = \frac{1}{2}\pi 1^2 = \frac{\pi}{2}.$$

Now we have the two funny wedges pieces.
Actually, the whole thing is symmetrical, so all we need do is work out the area of the blue wedge on the left and double it.

Hmm. What is that blue wedge?

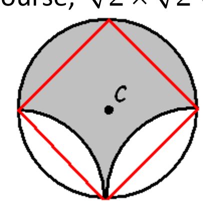
Look, I am just going to assume everything that looks 90° in this picture really is! (I bet it is.) The blue region is part of a square:



The area of the square is $1 \times 1 = 1$, and the unshaded portion of the square is a quarter-circle of area $\frac{1}{4}\pi 1^2 = \frac{\pi}{4}$ and so the blue shaded region has area: $1 - \pi/4$. This means the whole area we seek is:
$\pi/2 + 2 \times (1 - \pi/4) = 2$ square units!

Extension 1: What theorems in geometry do in fact show that every angle we assumed was 90° in measure really is so? How?

Extension 2: Stare at this picture to see that the area of the shaded region is, of course, $\sqrt{2} \times \sqrt{2} = 2$!



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