

Curriculum Burst 39: Skipped Counts

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Seven students count from 1 to 1000 as follows:

- Alice says all of the numbers, except she skips the middle number in each consecutive group of three numbers. That is, Alice says 1, 3, 4, 6, 7, 9, ..., 997, 999, 1000.
- Barbara says all of the numbers that Alice doesn't say, except she also skips the middle number in each consecutive group of three numbers.
- Candice says all of the numbers that neither Alice nor Barbara says, except she also skips the middle number in each consecutive group of three numbers.
- Debbie, Eliza, and Fatima say all of the numbers that none of the students with first names beginning before theirs in the alphabet say, except each also skips the middle number in each of her consecutive groups of three numbers.
- Finally, George says the only number that no one else says.

What number does George say?

THE QUICK STATS:

MAA AMC GRADE LEVEL

This question is appropriate for the 10th grade level.

MATHEMATICAL TOPIC

Integer exponents

COMMON CORE STATE STANDARDS

- **8.EE.A:** Work with integer and radical exponents.
- **A-SSE.2** Use the structure of an expression to identify ways to rewrite it.

MATHEMATICAL PRACTICE STANDARDS

- MP1 Make sense of problems and persevere in solving them.
- MP2 Reason abstractly and quantitatively.
- MP7 Look for and make use of structure.
- MP8 Look for and express regularity in repeated reasoning.

PROBLEM SOLVING STRATEGY

ESSAY 4: DRAW A PICTURE

SOURCE

This is question # 23 from the 2011 MAA AMC 10A Competition.



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THE PROBLEM-SOLVING PROCESS:

As always ...

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

This question is truly overwhelming. If the length of time it takes just to read the question is any indication of the length of time needed to solve it, then I am going to be here for quite a while!

One strategy for solving the question comes to mind right away: the brute force approach. We could write out all the numbers from 1 to 1000, cross out all the numbers Alice says, then all the numbers Barbara says, and so on. That is, we could just carry out the experiment and see what is left for George. But my reaction to that approach is Ugh! It seems like too much work.

But as I sit with this problem I really cannot think of what else to do.

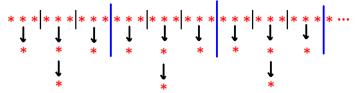
Alright then ... Is there a way to conduct the experiment on paper that cuts down on the work? I certainly don't have to write all the numbers from 1 to 1000. I could just write the first few numbers – the first twenty, say - and get a feel for the problem. And rather than write the actual numbers, what if I just drew a row of dots or stars to represent them?

Here's my quick representation of the first 1000 numbers:

Here's a quick visual representation of what Alice passes on to Barbara; the middle number of each set of three.



Okay. What does Barbara pass on to Candice? The middle number of each set of three of these:



And what does Candice pass on to Debbie? The middle of each set of three of these on the bottom row. And so on.

I think this picture is making things clear.

Barbara receives the middle number of every group of three among the numbers $1 \mbox{ to } 1000.$

Candice receives the middle number of every group of nine among the numbers $1 \mbox{ to } 1000 \,.$

Debbie will receive the middle number of every group of $27\,\rm{three}$ among the numbers $1\,\rm{to}\,1000\,.$

(We're dealing with the powers of three.) Let's keep going!

Eliza receives the middle number of every group of $81\,\mathrm{three}$ among the numbers 1 to $1000\,.$

Fatima receives the middle number of every group of $243\,\rm{three}$ among the numbers 1 to $1000\,.$

George receives the middle number of every group of 729 three among the numbers 1 to 1000.

What's the middle number of 1, 2, ..., 728, 729? And what's the middle number of 730, 731, ..., 1458? Hmm. I want the number halfway between 1 and 729. That would be $1 + \frac{729 - 1}{2} = 365$. And the number halfway between 730 and 1458 is $730 + \frac{728}{2} = 1094$.

Oops! Too big. George won't be given that number.

Oh! That's it. George is handed just a single number and that number is 365. (Wow! The power of a picture.)

Aside: What is the general formula for the middle number of a list of numbers a, a + 1, a + 2, ..., b - 1, b? (Is asking for the middle number of a list always meaningful?)

Extension: a) What if the girls skipped instead the third number of every group of three numbers?b) What if each girl randomly skipped a number from each set of three numbers. Is it possible for George to be left with the number 500? The number 1?

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