Curriculum Burst 50: A Curved Area

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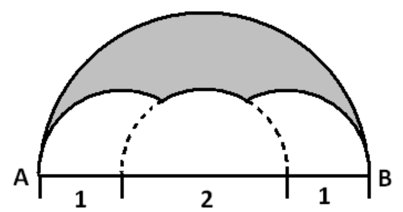
Three semicircles of radius 1 are constructed on diameter $\overline{AB}$ of a semicircle of radius 2. The centers of the small semicircles divide $\overline{AB}$ into four line segments of equal length, as shown. What is the area of the shaded region that lies within the large semicircle but outside the smaller semicircles?



(A) $\pi-\sqrt{3}$ (B) $\pi-\sqrt{2}$ (C) $\frac{\pi+\sqrt{2}}{2}$ (D) $\frac{\pi+\sqrt{3}}{2}$ (E) $\frac{7}{6}\pi-\frac{\sqrt{3}}{2}$

QUICK STATS:

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MAA AMC GRADE LEVEL

This question is appropriate for the 10th grade level.

MATHEMATICAL TOPICS

Geometry

COMMON CORE STATE STANDARDS

G-C.B: Find arc lengths and areas of sectors of circles.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 6: **ELIMINATE INCORRECT CHOICES**

SOURCE: This is question # 19 from the 2003 MAA AMC 10b Competition.

THE PROBLEM-SOLVING PROCESS:

The appropriate first step is …

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STEP 1: Read the question, have an
emotional reaction to it, take a deep
breath, and then reread the question.
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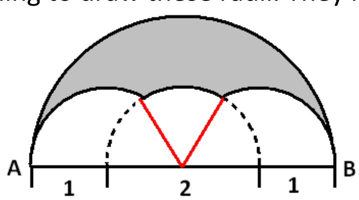
I look at this question and know I should just dive right in - find the areas of the semicircles and begin adding and subtracting areas. No doubt there is going to be some clever observation I'll need to make along the way, just as a twist, but this overall approach feels manageable.

But given that I've been presented with five options to select from, let me try something else here. Let's focus on the answers offered.

I can't help but notice the $\sqrt{2}$ s and $\sqrt{3}$ s presented in them. How could the answer to this problem possibly involve a square root of two or a square root of three?

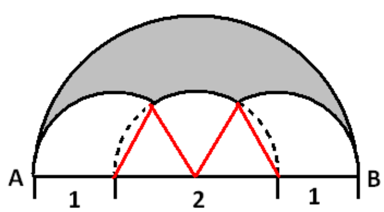
I know that $\sqrt{2}$ s appear when I think of $45-90-45$ triangles (half squares) and $\sqrt{3}$ s in $30-60-90$ triangles (half equilateral triangles). Are there half squares or equilateral triangles in this problem?

It feels compelling to draw these radii. They have length 1.



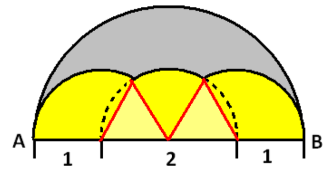
This gives a sector with angle 60°. That's close to an equilateral triangle. Hmm.

Actually, let me draw two more lines:



There are lots of 60° angles in this picture. The two triangles I see are isosceles triangles (the sides that are radii of length 1) containing 60° angles. They are actually equilateral triangles –with side length 1. Okay. The answer is going to involve $\sqrt{3}$ s.

Let's go further …

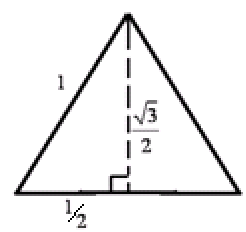


The shaded region is a semi-circle with three yellow sectors of a circle removed and two equilateral triangles removed. The answer must be of the form:

Something involving π - something involving $\sqrt{3}$

So it's (A) or (E).

The area of an equilateral triangle of side length 1 is $\frac{1}{2}\times 1\times\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{4}$.



(The Pythagorean Theorem tells me its height.) As there are two of these triangles, the answer must be of the form:

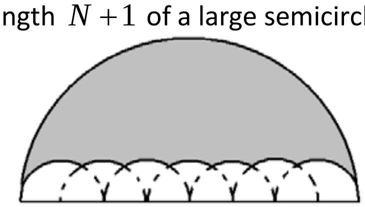
Something involving π $-\frac{\sqrt{3}}{2}$

The answer can only be (E).

Question: Actually can you <u>see</u> the answer:

$$\frac{1}{2}\cdot\pi 2^2-\frac{1}{3}\cdot\pi 1^2-\frac{1}{6}\cdot\pi 1^2-\frac{1}{3}\cdot\pi 1^2-\frac{\sqrt{3}}{2}\ ?$$

Extension: N semicircles of radius 1 are stacked along a diameter of length $N+1$ of a large semicircle.



What is the area of the shaded region in terms of N ?

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