Curriculum Burst 52: Melted Ice Cream

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> An ice cream cone consists of a sphere of vanilla ice cream and a right circular cone that has the same diameter as the sphere. If the ice cream melts, it will exactly fill the cone. Assume that the melted ice cream occupies 75% of the volume of the frozen ice cream. What is the ratio of the cone's height to its radius?
> (Note: A cone with radius r and height h has volume $\pi r^2 h / 3$, and a sphere with radius r has volume $4\pi r^3 / 3$.)

QUICK STATS:

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MAA AMC GRADE LEVEL

This question is appropriate for the 10th grade level.

MATHEMATICAL TOPICS

Geometry

COMMON CORE STATE STANDARDS

G-GMD.A.3   Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

MATHEMATICAL PRACTICE STANDARDS

MP1   Make sense of problems and persevere in solving them.
MP2   Reason abstractly and quantitatively.
MP3   Construct viable arguments and critique the reasoning of others.

PROBLEM SOLVING STRATEGY

ESSAY 8:   **SECOND GUESS THE AUTHOR**

**SOURCE:** This is question # 17 from the 2003 MAA AMC 10b Competition.
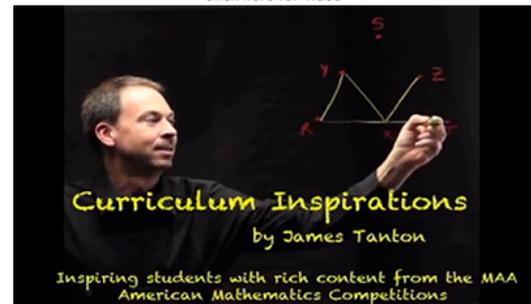
THE PROBLEM-SOLVING PROCESS:

As always, the first step is …

> **STEP 1:** Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

As I read this question I am confused! We have a sphere of vanilla ice-cream sitting in a circular cone and the diameters of the sphere and the cone match.
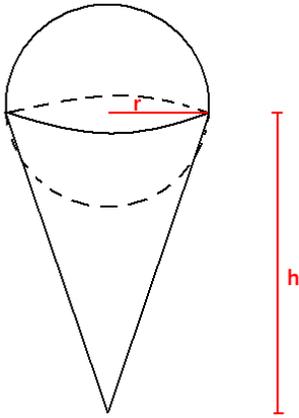
(Hmm. I am wondering as I draw this: Does the rim of the cone sit right on the "equator" of the sphere? My picture has some ice-cream bulging though the cone. But maybe that is just my picture.)

We are told that if the ice-cream melts it will exactly fill the cone. But it is the next sentence that throws me:

> *Assume that the melted ice cream occupies 75% of the volume of the frozen ice cream.*

What does that mean? Does this ball of ice-cream have some melt in it? Is 75% of it already melted? If it were a sloppy mixture of frozen solid and liquid melt, how would it hold its shape as a ball of ice-cream?

Maybe the interior is melted and the outer rim is solid and holding it all together?

But if there were something physically strange going on in this question surely the author would have made mention of it. Since she/he didn't, I must be misunderstanding something.

What else can "Assume that the melted ice cream occupies 75% of the volume of the frozen ice cream." mean?

Alright … let's go back a step. What do I know in general about frozen things and melted things? I know ice floats. (It's a start!) And ice floats because solid water is less dense than the liquid water. Hmmm.

So a given mass of water must occupy more volume as ice as it does as water. That is, melted ice has less volume than solid ice!

Okay .. That's what this question must be saying!

> *Assume that the melted ice cream occupies 75% of the volume of the frozen ice cream.*

As liquid, the volume of the melt is three-quarters the volume of the sphere: $\frac{3}{4}\left(\frac{4}{3}\pi r^3\right) = \pi r^3$. This volume of this liquid equals the volume of the cone:

$$\pi r^3 = \frac{1}{3}\pi r^2 h.$$

This gives $r = \frac{1}{3}h$ or $h = 3r$. The height to the radius come in a three-to-one ratio. Done!

> **Extension:** So … Can a sphere of radius r sit perfectly inside a cone of radius r? Do the cone and the sphere meet at the equator of the sphere?

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