Curriculum Burst 78: How Many Twos?

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> For $k > 0$, let $I_k = 10\cdots064$, where there are k zeros between the 1 and the 6. Let $N(k)$ be the number of factors of 2 in the prime factorization of I_k. What is the maximum value of $N(k)$?

QUICK STATS:

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MAA AMC GRADE LEVEL

This question is appropriate for the lower high-school grade levels.

MATHEMATICAL TOPICS

Factors and Primes, Integer Exponents

COMMON CORE STATE STANDARDS

A.SEE.1b Interpret complicated expressions by viewing one or more of their parts as a single entity.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 10: [GO TO EXTREMES]

SOURCE: This is question # 25 from the 2009 MAA AMC 10A Competition.

THE PROBLEM-SOLVING PROCESS:

As always, the best start is …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

This question looks positively scary! It's going to take some doing just to understand what is being asked.

So we have all these numbers of the form "$10\cdots0064$" –a one, some zeros, and then a six and a four. Just to be clear I am going to write them out, catching the very beginning and "end" cases:

164 (no zeros)
1064
10064
100064

and

$10000000000000000000000064$

and so on.

The "$N(k)$" part of the question is scary, so I am going to see if I can get away with ignoring it.

I need to look at the prime factorizations of these numbers and count the number of 2 s in those factorizations.

Hmm. There is no way I am going to factor these numbers! But how can I tell how many 2 s are in the factorization?

Look at the smallest number: How many 2 s are in the prime factorization of 164? Well, I can divide 164 by 2 two times: $164 \to 82 \to 41$.This tells me that there are two 2 s in its prime factorization: $164 = 2^2 \times \textit{other primes}$.

This problem is starting to feel do-able.

In rereading the question I see that this $N(k)$ business is something about the count of 2 s in all these prime factorizations, and we want the biggest this number can be. The question basically is: *What is the largest count of 2 s you can get from these prime factorizations?*

Well, we did the work for one extreme on the list, the number 164. Let's do it for something at the other extreme, just to get more of a feel for things.

Let's divide 1000000000064 (ten zeros) by 2 as many times as we can.

$$
\begin{aligned}
&1\ 000\ 000\ 000\ 064\\
&\to 500\ 000\ 000\ 032\\
&\to 250\ 000\ 000\ 016\\
&\to 125\ 000\ 000\ 008\\
&\to 62\ 500\ 000\ 004\\
&\to 31\ 250\ 000\ 002\\
&\to 15\ 625\ 000\ 001
\end{aligned}
$$

Okay. Six times. So with lots of zeros involved there are six 2 s in the prime factorization. Can we beat six?

Actually, in the above work there we left with five middle 0 s. So if we took away five zeros from the ten we began with, we'd still be able to divide by 2 six times (and no more).

What happens with one, two, three or four 0 s? Let's look at the number 1000064 next. The arithmetic above stays the same, except after the fifth division by 2 we have: $31252 = 31250 + 2$. Dividing by two one more time gives: $15625 + 1 = 15626$, which can be divided by 2 one more time:

$15626 \to$ *something that ends with* 3.

Thus 1000064 has <u>seven</u> 2 s in its prime factorization.

Alright, how about three zeros? I am getting tired of the arithmetic, how about thinking of this as: $100064 = 100000 + 64 = 10 \times 10 \times 10 \times 10 \times 10 + 64$?

Dividing this by 2 five times gives: $5 \times 5 \times 5 \times 5 \times 5 + 2$, which is an odd number and so cannot be divided in half a sixth time. Actually… $10064 = 10 \times 10 \times 10 \times 10 + 64$ can only be halved four times, $1064 = 10 \times 10 \times 10 + 64$ three times, and $164 = 10 \times 10 + 64$ twice, as we've already seen. (I like this approach of think of products of 10 plus 64. I wish I thought of this approach first!)

We have that the largest number of 2 s among the prime factorizations of these numbers is <u>seven</u> (for 1000064).

Extension: What is the largest count of 2 s that appear among the prime factorizations of the numbers of the form $10^k + 2^{100}$?

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