Curriculum Burst 129: A Magic Square

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In the magic square shown, the sums of the numbers in each row, column, and diagonal are the same.

v	24	w
18	x	y
25	z	21

Five of the numbers are represented by v, w, x, y, *and* z. *What is the value of* $y+z$ *?*

QUICK STATS:

MAA AMC GRADE LEVEL

This question is appropriate for the lower high-school grades.

MATHEMATICAL TOPICS

Algebra: simultaneous linear equations

COMMON CORE STATE STANDARDS

A-REI.6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 10: **GO TO EXTREMES**

SOURCE: This is question # 22 from the 2001 MAA AMC 10 Competition.

THE PROBLEM-SOLVING PROCESS:

The best, and most appropriate, first step is always …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

Every row, every column, and every diagonal of the magic square has an unknown value in it. This means I don't know what the magic sum of the square is! That makes the question feel hard.

I do notice that the question is not asking me for an actual value of one of the unknowns, only the value of the sum of two of them. I suspect this means I won't be able to work out the value of each unknown. Hmm.

Well there are three rows, three columns, and two diagonals. This makes for eight triples that all have the same sum. Should I call that sum M, bring in yet another symbol for an unknown?

We have:

$$v+24+w=M$$
$$18+x+y=M$$
$$25+z+21=M$$
$$v+18+25=M$$

and so on. (I really don't feel like writing them all out.)

BUT … I do notice that some of these equations have more numbers than they do variables. Perhaps I should focus on those equations. They might give the most information:

$$25+z+21=M \Rightarrow z=M-46$$
$$v+18+25=M \Rightarrow v=M-43$$

M-43	24	w
18	x	y
25	M-46	21

Now is seems like x and w are parts of equations with the most information:

$$24+x+M-46=M \Rightarrow x=22$$
$$M-43+24+w=M \Rightarrow w=19$$

Ooh. Actual values!

M-43	24	19
18	22	y
25	M-46	21

Now I see $19+y+21=M \Rightarrow y=M-40$.

M-43	24	19
18	22	M-40
25	M-46	21

So $y+z=(M-40)+(M-46)=2M-86$. But I don't know what M is. Oh! Look at the northeast diagonal:

$$25+22+19=M \Rightarrow M=66.$$

So $y+z=2\times 66-86=46$. (If I noticed that diagonal earlier, I could have gotten to this a step or two ago. No worries, though.)

The answer is 46.

(Notice that actually we have enough information to work out the exact value of each of the unknowns after all!)

Extension: Each cell of a 15×17 grid contains a non-negative whole number. Each row and each column of the grid sum to the same magic value M. What is the value of M?

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