Curriculum Inspirations
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MAA

Curriculum Burst 136: An Integer Sequence

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Let $\{a_k\}$ be a sequence of integers such that $a_1 = 1$ and $a_{m+n} = a_m + a_n + mn$, for all positive integers m and n. What is a_{12}?

QUICK STATS:

MAA AMC GRADE LEVEL

This question is appropriate for the lower high-school grades.

MATHEMATICAL TOPICS

Sequences

COMMON CORE STATE STANDARDS

F-IF.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 1: **ENGAGE IN SUCCESSFUL FLAILING!**

SOURCE: This is question # 23 from the 2002 MAA AMC 10B Competition.

THE PROBLEM-SOLVING PROCESS:

The best, and most appropriate, first step is always …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

This question looks strange. We have a sequence of integers a_1, a_2, a_3, … with $a_1 = 1$. And then, thereafter, $a_{m+n} = a_m + a_n + mn$. Hmm.

Well, to get a feel for things, this means for example,

$$a_5 = a_2 + a_3 + 2 \times 3 = a_2 + a_3 + 6$$

and

$$a_2 = a_1 + a_1 + 1 \times 1 = 1 + 1 + 1 = 3.$$

Oh, I can build up the sequence:

$$a_3 = a_2 + a_1 + 2 = 3 + 1 + 2 = 6$$

and

$$a_4 = a_3 + a_1 + 3 = 6 + 1 + 3 = 10$$

and so on. I could keep plugging away up to a_{12}.

OR … I could just use:

$$a_{12} = a_6 + a_6 + 36$$

and

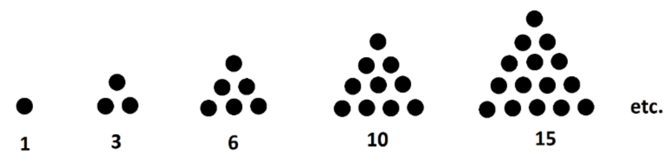
$$a_6 = a_3 + a_3 + 9 = 6 + 6 + 9 = 21.$$

Putting this back into a_{12} we get:

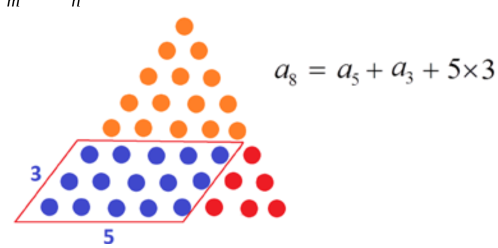
$$a_{12} = 21 + 21 + 36 = 78.$$

Wow! Swift!

Extension: If we keep plugging away at the terms of the sequence we get: $a_1 = 1$, $a_2 = 3$, $a_3 = 6$, $a_4 = 10$, $a_5 = 15$, $a_6 = 21$, and so on. These are the triangle numbers:



Do you see how the following picture shows that these triangle numbers satisfy the relation $a_{m+n} = a_m + a_n + m \times n$:



Another sequence of integers $\{b_k\}$ with $b_k = 1$ satisfies $b_{m+n} = b_m + b_n + 2mn$ for all positive integers m and n. Can you see what geometric numbers these are? Do you see why these numbers satisfy this relation?

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