Curriculum Burst 145: Rounded Roots

By Dr. James Tanton, MAA Mathematician at Large

> The symbolism $\lfloor x \rfloor$ denotes the largest integer not exceeding x. For example, $\lfloor 3 \rfloor = 3$ and $\lfloor 9/2 \rfloor = 4$.
> What is $\lfloor \sqrt{1} \rfloor + \lfloor \sqrt{2} \rfloor + \lfloor \sqrt{3} \rfloor + \cdots + \lfloor \sqrt{16} \rfloor$?

QUICK STATS:

MAA AMC GRADE LEVEL

This question is appropriate for the lower high-school grades.

MATHEMATICAL TOPICS

Algebra and Pre-Calculus: Summation

COMMON CORE STATE STANDARDS

F-IF.3 (Tangentially) Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 7: [Perseverance is key.]

SOURCE: This is question # 7 from the 2003 MAA AMC 10B Competition.

THE PROBLEM-SOLVING PROCESS:

The best, and most appropriate, first step is always …

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STEP 1: Read the question, have an
emotional reaction to it, take a deep
breath, and then reread the question.
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The symbolism is new to me and I need to think through it slowly.

$\lfloor x \rfloor$ *denotes the largest integer not exceeding* x.

"Exceeding" means "not bigger than."

We have to work out the sum of sixteen values. The first is:

$\lfloor \sqrt{1} \rfloor = \lfloor 1 \rfloor = 1$, since 1 is the largest integer not bigger than 1.

$\lfloor \sqrt{2} \rfloor = \lfloor 1.414.... \rfloor = 1$, since 1 is the largest integer not bigger than $1.414....$.

$\lfloor \sqrt{3} \rfloor = \lfloor 1.7... \rfloor$ (or is $\sqrt{3} \approx 1.8$? I can't remember!) In any case, $\lfloor \sqrt{3} \rfloor = 1$, since 1 is the largest integer not bigger than $1.7...$ or $1.8....$.

$\lfloor \sqrt{4} \rfloor = \lfloor 2 \rfloor = 2$.

Ahh! $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, and $\sqrt{8}$ are all between 2 and 3 and so

$$\lfloor \sqrt{5} \rfloor = \lfloor \sqrt{6} \rfloor = \lfloor \sqrt{7} \rfloor = \lfloor \sqrt{8} \rfloor = 2.$$

And I see:

$$\lfloor \sqrt{9} \rfloor = \lfloor \sqrt{10} \rfloor = \cdots = \lfloor \sqrt{15} \rfloor = 3$$

and finally:

$$\lfloor \sqrt{16} \rfloor = 4.$$

So the sum we seek is:

$$\begin{aligned} &1+1+1+2+2+2+2+2 \\ &\quad +3+3+3+3+3+3+3+4 \\ &= 3+10+21+4 \\ &= 38. \end{aligned}$$

Great!

Extension 1: Look at the sequence of numbers $\lfloor \sqrt{1} \rfloor, \lfloor \sqrt{2} \rfloor, \lfloor \sqrt{3} \rfloor, \lfloor \sqrt{4} \rfloor, \lfloor \sqrt{5} \rfloor, \lfloor \sqrt{6} \rfloor, ...$. How many numbers in this sequence have the value one million?

Extension 2: Is there a formula for the sum $\lfloor \sqrt{1} \rfloor + \lfloor \sqrt{2} \rfloor + \lfloor \sqrt{3} \rfloor + \cdots + \lfloor \sqrt{n^2 - 1} \rfloor$?

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