Problem Solving Strategy Essay # 1:

Engage in Successful Flailing

James Tanton, PhD, Mathematics, Princeton 1994; MAA Mathematician in Residence

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Teachers and schools can benefit from the chance to challenge students with interesting mathematical questions that are aligned with curriculum standards at all levels of difficulty.

This is the first essay in a series to give credence to this claim on the MAA website, www.maa.org/math-competitions. For over six decades, the MAA AMC has been creating and sharing marvelous stand-alone mathematical tidbits. Take them out of their competition coverings and see opportunity after opportunity to engage in great conversation with your students. Everyone can revel in the true creative mathematical experience!

I personally believe that the ultimate goal of the mathematics curriculum is to teach self-reliant thinking, critical questioning and the confidence to synthesize ideas and to re-evaluate them. Content, of course, is itself important, but content linked to thinking is the key. Our complex society is demanding of the next generation not only mastery of quantitative skills, but also the confidence to ask new questions, explore, wonder, flail, innovate and succeed. Welcome to these essays!

We will demonstrate the power of mulling on interesting mathematics and develop the art of asking questions. We will help foster good problem solving skills, and joyfully reinforce ideas and methods from the mathematics curriculum. We will be explicit about links with goals of the Common Core State Standards for Mathematics. Above all, we will show that deep and rich thinking of mathematics can be just plain fun!

So on that note … Let's get started!

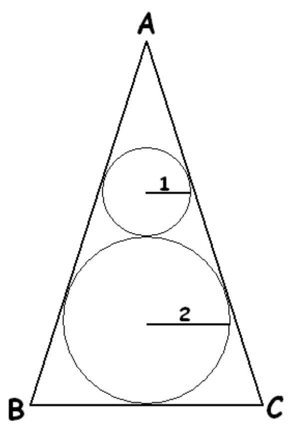
CONTENTS: Here we shall:

a. Present a problem
b. Discuss problem-solving methods in the context of solving that problem.
c. Explicitly connect the problem to the curriculum, the Core Content Standards and the Mathematical Practices.
d. Take the problem further.
e. Explore deeper curricular mathematics inspired by the problem.

OUR CHALLENGE TODAY

This month's challenge is Question 3 from the AMC 10/12 Practice Quiz on "Geometry of Triangles, II."

A circle of radius 1 ***is tangent to a circle of radius*** 2***. The sides of*** ΔABC ***are tangent to the circles as shown, and the sides*** $\overline{AB}$ ***and*** $\overline{AC}$ ***are congruent. What is the area of*** ΔABC***?***



TEACHING PROBLEM SOLVING – and solving the problem!

Here is step one to the art of solving a problem:

STEP 1: **Take a deep breath and relax!**

Students (and adults too!) are often under the impression that one should simply be able to leap into a mathematics challenge and make instantaneous progress of some kind. This not how mathematics works! It is okay to fumble, and flail, and try ideas that turn out not help in the end. This is a natural part of problem solving that should not at all be dismissed! (This, after all, is the beginning of the process of how we solve problems in life.)

Once we are comfortable knowing we are sure to fumble and flail, go to step two to see if we can at least engage in "organized flailing."

STEP 2: **Ask, "What is this problem about?" What do I know that might be relevant to its theme?**

For starters, this seems to be a geometry problem. (Stating the obvious is very good way to get going!) It seems to be about:

- Circles
- Triangles
- Areas of Triangles
- Tangent lines to circles

There might be more involved, but these are the big concepts that leap out.

We can go a step further still and organize ideas in more detail.

STEP 3: What facts on these topics do I know?

The advantage of looking at a problem like this in the classroom is that you can introduce it a time appropriate to the content just covered. If we share this problem with our students in a second-semester geometry course, students will naturally begin to list results that might be relevant:

Facts about tangent lines to circles:

- Two tangents to a circle from a common point are congruent. (G-C-2)
- A radius and a tangent are perpendicular at the point of contact. (G-C-2)

Facts about triangles:

- Similarity and congruence results. (G-CO-8, G-SRT-3)
- Right triangles and Pythagoras's Theorem (8-G-B, G-SRT-4)
- Area formula (6-G-1)

Facts about circles in triangles:

- The construction of incircles. (G-C-3).

Here are some short videos explaining the mathematics of these results. (The properties of "incircles," for example, has not been a standard curriculum topic in past decades.)

Tangent Lines to Circles: www.jamestanton.com/?p=1230
Incircles and Circumcircles: www.jamestanton.com/?p=878
Pythagoras's Theorem: www.jamestanton.com/?p=1219

Students might enjoy watching these videos too.

With lists in hand we are now feeling set to work on the problem!

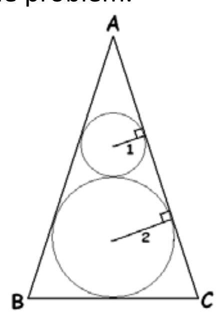
The next step in problem solving is often a surprise to students!

> STEP 4. **No one says you must work with the diagram presented to you! Feel free to redraw and eliminate features, or to draw in extra features** - *as long as they don't change the question!*

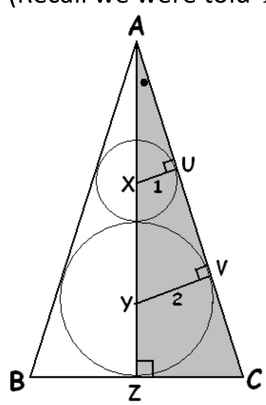
But this step comes with an extra caveat …

> **4. (Continued) … But don't draw in too much extra content! Try drawing in at most one or two extra lines, if any at all.** *That is, don't make the picture more complicated: make it simpler.*

Given that my mind is thinking about "perpendicular radii and tangents," maybe it would be good to redraw the two radii presented in the problem.



Does it feel compelling now to draw the altitude of the big isosceles triangle? (Recall we were told $\overline{AB} \cong \overline{AC}$.)



I have labeled points X, Y, Z, U and V along the way.

We are trying to find the area of ΔABC, but finding the area of the shaded half, ΔAZC will suffice. (Just double the result.) And for that we need to know the height of this triangle and its base-length. Right now we know neither!

By the way … This illustrates a good problem solving strategy: *Can you identify the "penultimate step"? Ask: What do I need to know that would then give the result I seek?* We need to know the lengths ZC and AZ.

Let's go back to steps 2 and 3, for this more refined sub-problem: What is "Finding the lengths ZC and AZ about?" and "What facts do I know about this type of task?"

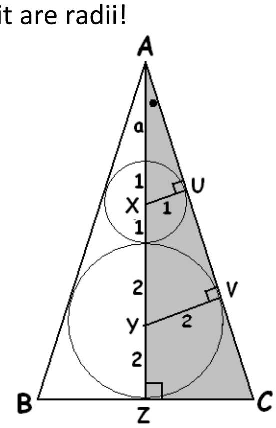
Well, ZC and AZ are side-lengths in right triangles and I know two general methods for finding such triangle side-lengths:

- Pythagoras's theorem.
- Matching sides in similar triangles.

We have lots of right triangles, well, three: ΔAUX and ΔAVY and ΔAZC, and they are all similar to each other by the AA principle. (Whoa! They each possess a right angle and share the angle labeled "dot.") I am not sure where this going, but I know ΔAUX and ΔAVY are similar with scale factor two ($YV = 2$ and $XU = 1$) and so $AY = 2 \cdot AX$. This feels like it is getting at working out the height of the triangle. (Perhaps no need for Pythagoras?)

Much of problem solving relies on waiting for an epiphany. By systematically laying out what you are heading for and what you already know, you significantly increase your chances for one.

Epiphanies, unfortunately, do not come on command and will take their own good time. And that's okay! Maybe it is time to put the problem aside and go for a walk? However, if you are leading this problem as a class discussion, I wouldn't be surprised if someone in the room noticed that we already know a great deal about the length AZ: sections of it are radii!



The only portion of AZ we do not know is labeled a. But from similar triangles we have $AY / AX = 2$, that is, $\frac{a+4}{a+1} = 2$, giving $a = 2$ and $AZ = 8$!

Can we repeat the trick for ZC?

$\Delta AUX \sim \Delta AZC$, but they are similar with a twist: side $UX = 1$ matches with ZC, but it is AU that matches with $AZ = 8$. But Pythagoras tells me $AU = \sqrt{9-1} = \sqrt{8}$. We're good!

From $\frac{ZC}{UX} = \frac{AZ}{AU}$ we get $\frac{ZC}{1} = \frac{8}{\sqrt{8}}$ and so $ZC = \sqrt{8}$.

The area of ΔABC is thus $2 \times \frac{1}{2} \cdot ZC \cdot AZ = 8\sqrt{8} = 16\sqrt{2}$.

Phew!

COMMON CORE STANDARDS and PRACTICES:

This AMC problem is connected to the CCSS-M content standards previously outlined. But, more important, being very explicit and clear with students about the process of problem solving by following a discussion like the one I've modeled here hits right on the mark of the following practice standards:

MP1: Make sense of problems and persevere in solving them.
MP2: Reason abstractly and quantitatively.
MP3: Construct viable arguments and critique the reasoning of others.
MP5: Use appropriate tools strategically.
MP7: Look for and make use of structure.

But we can, and should, go further …

DECONSTRUCTING THE PROBLEM:

A great sense of accomplishment comes from solving a challenge. Good for us! But true progress and innovation in science and business comes from pushing boundaries, asking new questions, and forging interesting paths of one's own devising. Let's help our next generation of citizens do that too!

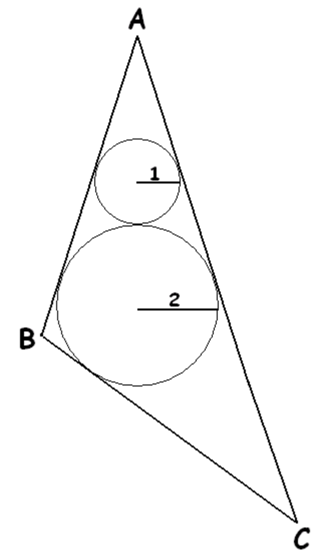
Here's the problem again:

A circle of radius 1 is tangent to a circle of radius 2. The sides of ΔABC are tangent to the circles as shown, and the sides $\overline{AB}$ and $\overline{AC}$ are congruent. What is the area of ΔABC?

How can we use this problem to inspire original enquiry and discovery?

Idea 1: Challenge the problem writer.

Why did the writer feel it necessary to insert condition that $\overline{AB}$ and $\overline{AC}$ are congruent? Must ΔABC be isosceles?



Where did we use the fact that ΔABC was isosceles in our solution?

Computer Project: Using a geometry software package, analyze the areas of triangles that circumscribe two tangent circles as shown above. (Don't worry about choosing radii 1 and 2.) Keep the point A and the two circles fixed but vary the tangent line segment $\overline{BC}$. Could the area of the triangle be arbitrarily large? Which type of triangle gives the smallest area? Can you prove true any observations you make?

Idea 2: Generalize the problem.

What is the area of the isosceles triangle ΔABC if the small circle has radius r_1 and the larger one r_2? Is there are lovely way to express the formula for that area? (What has "$16\sqrt{2}$" to do with the original numbers 1 and 2?)

Idea 3: How special is the problem?

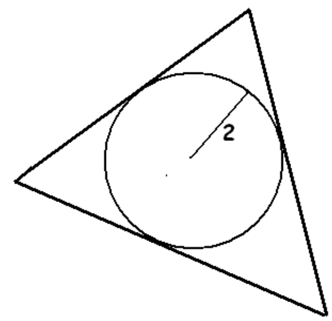
We know that for any given triangle we can draw a circle tangent to all three sides. This is the *incircle* of the triangle. (See the video Incircles and Circumcircles.) The circle of radius 2 in the original problem is the incircle of ΔABC. BUT … Is it always possible to draw a second circle, like the circle of radius 1 in the problem, inside a triangle tangent to its incircle and to two of the sides? Hmm! What do you think?

Idea 4: Make use of your own special knowledge.

I have an advantage in having thought about incircles before. I know that there is something special about incircles of radius two. This problem made me think of it.

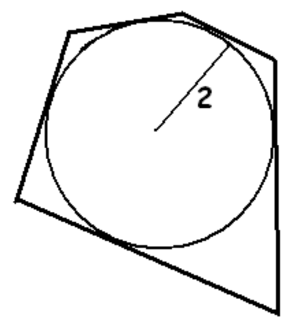
Computer Project: AREA/PERIMETER SURPRISE

1. Using a geometry software package draw a circle of radius 2 and construct a triangle with three sides tangent to it.



Compute the area of the triangle and the perimeter of the triangle. What astounding thing do you notice? Repeat this work for other triangles circumscribing the circle just to make sure that this isn't just coincidence!

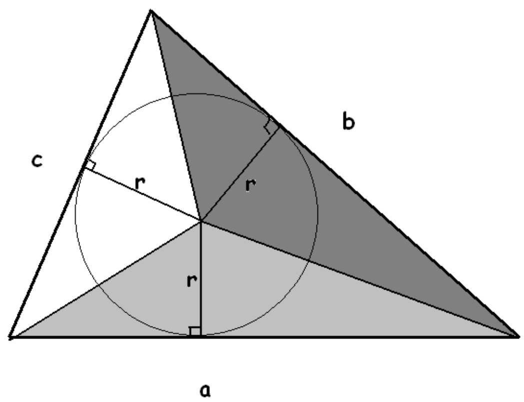
2. Why stop at triangles? Draw a polygon with sides tangent to a circle of radius 2. What do you notice about its perimeter and area?



3. Why is what you are observing true? What is special about the number two here?

Just to give things away …

Suppose a triangle with side-lengths a, b and c circumscribes a circle of radius r.



Dividing the large triangle into three sub-triangles each of height r we see that its area A is given by:

$$\begin{aligned} A &= \frac{1}{2}\cdot a\cdot r+\frac{1}{2}\cdot b\cdot r+\frac{1}{2}\cdot c\cdot r \\ &= \frac{1}{2}(a+b+c)r \\ &= \frac{r}{2}\cdot P \end{aligned}$$

where $P=a+b+c$ is the perimeter of the triangle. In the special case with $r=2$ we have that $A=P$, the area and the perimeter have the same numerical value!

And the same is true for any polygon that circumscribes a circle of radius two. (Exactly the same argument works.)

Idea 5: Revisit the problem with new knowledge.

Now that we know that triangle ΔABC has area $16\sqrt{2}$, and that this is the same as its perimeter(!), could we have solved problem by computing the perimeter of the triangle instead? Would focusing on perimeter lead to a different, perhaps, "better" solution?

Mathematics is an ongoing discussion. Textbook questions, classroom discussions, and competition problems are not closed, finite experiences. They are invitations for conversation, exploration and further discovery. Let's keep open that door of conversation for our students.

Curriculum Inspirations is brought to you by the Mathematical Association of America and the MAA American Mathematics Competitions.