Pictures, Probability, and Paradox

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A wise old mathematician (probably) once said, "A mathematical picture is worth a thousand mathematical words." The solutions to many seemingly difficult probability problems are often "easy to see" when the right picture is drawn. Problems which can be solved pictorially can be useful in the general development of geometric intuition and in developing some of the basic ideas of probability. Some of the problems presented here will require the use of integral calculus, but many similar problems can be devised that do not require calculus. The last problem poses a paradox, for it is solved by three different methods: three different answers emerge, and all three are correct!

Problem 1. Two numbers are chosen at random between 0 and 10. What is the probability that their product is less than 25?

For some this problem has little to motivate it other than a curiosity which some students do not possess. A student's curiosity may be raised by asking him to make a guess at the answer. Having made a guess, he now has a vested interest in the solution, as has been pointed out by Pólya. Now that you have made your guess, we will proceed with the solution.

Let x and y represent the two numbers. Random selection means that any number between zero and ten has an equal chance of being selected. All the possible selections of x and y can be represented by the square in Figure 1. Each point of the square represents a pair of numbers which might be chosen, and each point (number pair) has an equal chance of being selected. So now, instead of thinking about pairs of numbers, we can think in terms of points in the square.

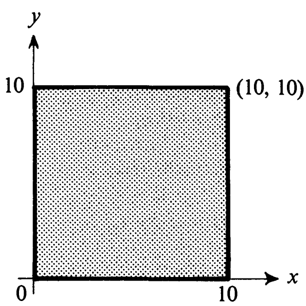

Figure 1.

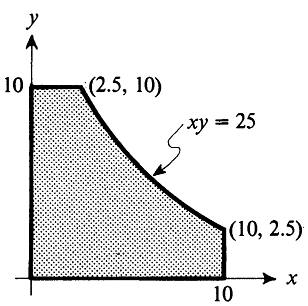

Figure 2.

We are particularly concerned with those points in the square that represent number pairs whose product is less than 25, i.e., those points in the square below the hyperbola $xy = 25$. (See Figure 2.)

It is now easy to "see" that the solution of this problem is simply equal to the ratio of the area in Figure 2 to the area of the square. Thus we obtain for our answer:

$$\frac{\text{Area of shaded area}}{\text{Area of the square}} = \frac{25 + \int_{2.5}^{10} \frac{25}{x}\,dx}{100} = \frac{25(1 + \ln 4)}{100} \approx 0.59657.$$

Using a computer to randomly select 100,000 points, we found the probability to be 0.59560.

Problem 2. Suppose a stick is cut at two random points into three segments. What is the probability that these three segments will form a triangle if placed end to end?

Solution. We will suppose our stick to be a unit line segment and represent their lengths by x, y, and $1 - x - y$. Since each of these segments must be greater than zero and less than one, the following inequalities result:

$$0 < x < 1, \qquad 0 < y < 1, \quad \text{and} \quad 0 < 1 - x - y < 1.$$

These conditions describe the triangular region S in Figure 3.

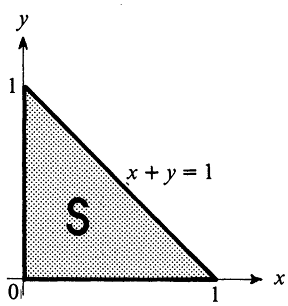

Figure 3.

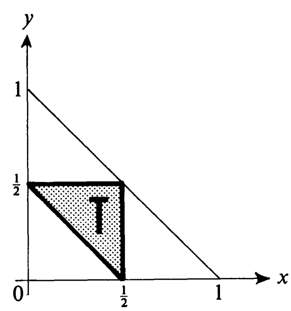

Figure 4.

Each point of S represents a possible partitioning of the stick into three parts. Now which of these partitionings will give us a triangle, i.e., which points of S represent a possible triangle?

A triangle can be formed if the sum of each pair of sides is greater than the third side. Since we started with a unit segment, this is equivalent to saying that each segment must be less than one-half. So if a triangle is to be formed, we must have

$$0 < x < 1/2, \qquad 0 < y < 1/2, \quad \text{and} \quad 0 < 1 - x - y < 1/2.$$

The points satisfying these conditions are shaded in Figure 4. We can now see that the answer to our problem is

$$\frac{\text{area } T}{\text{area } S} = \frac{1}{4}.$$

The next problem is an extension of the one we just completed.

Problem 3. If a stick is broken at random at two points, and if a triangle can be formed, what is the probability that the triangle is obtuse?

Solution. The problem now is to find which points of set T represent obtuse triangles. We will call this set of points set O. Using the fact that a triangle is obtuse if the square of any side is greater than the sum of the squares of the other two sides, the following conditions appear:

Angle A is obtuse if $x^2 > y^2 + (1 - x - y)^2$,

angle B is obtuse if $y^2 > (1 - x - y)^2 + x^2$, and

angle C is obtuse if $(1 - x - y)^2 > x^2 + y^2$.

These conditions simplify to

$$y < 1 - \frac{1}{2(1-x)}, \quad y > -x + \frac{1}{2(1-x)}, \quad \text{and} \quad x > -y + \frac{1}{2(1-y)}.$$

The above conditions combined with the conditions for set T define set O shaded in Figure 5.

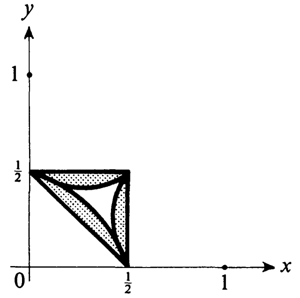

Figure 5.

We calculate the area of one of the three shaded parts and use a symmetry argument to claim the other two parts to have equal areas. If symmetry arguments are not convincing, you may wish to evaluate each of the necessary integrals. As you do more of these integrations, arguments of symmetry often become more appealing. The area of the upper shaded region equals

$$\int_0^{\frac{1}{2}} \left[\frac{1}{2} - \left(-x + \frac{1}{2(1-x)} \right) \right] dx = \frac{3 - 4 \ln 2}{8}.$$

Hence, the area of set $O = (9 - 12 \ln 2)/8$, and the probability that the triangle is obtuse is

$$\frac{\text{area } O}{\text{area } T} = 9 - 12 \ln 2 \approx 0.68223.$$

After 100,000 random selections, the computer got 0.68667.

Problem 4. A line segment is cut at three random points to form four segments. What is the probability that these four segments can be joined end to end to form a quadrilateral?

Solution. Using the technique of the previous problem, we will call the lengths of the segments x, y, z, and $1 - x - y - z$. We must have $0 < x < 1$, $0 < y < 1$, $0 < z < 1$ and $0 < 1 - x - y - z < 1$. The picture satisfying these conditions is the pyramid in Figure 6. Every point in pyramid $OABC$ represents a different partitioning of the segment, and each of these partitionings (points) has an equal likelihood of being selected.

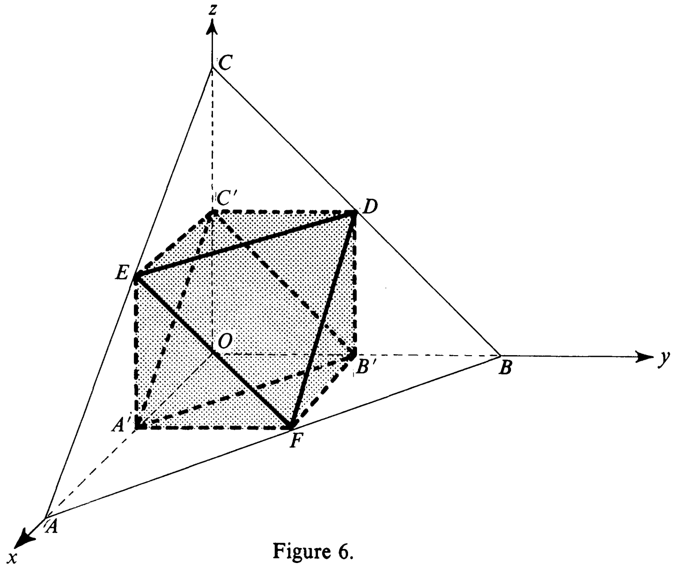

Figure 6.

Let S be the set of points that represent a successful partitioning of the line, i.e., a partitioning that will form a quadrilateral. Let F be the set of points that represent failures. Since a quadrilateral is only possible when each segment is less than one-half, we seek the set of points in the large pyramid P that also satisfy the inequalities $0 < x < 1/2$, $0 < y < 1/2$, $0 < z < 1/2$, and $0 < 1 - x - y - z < 1/2$. S is the set of points in the octahedron $A'FB'DC'E$. Hence the probability that a quadrilateral can be formed equals (volume of S)/(volume of P). The volume of $P = (1/3)(1/2)(1) = 1/6$. The volume of S can be found by subtracting the volume of F from P. Now the volume of F is $4(1/3)(1/8)(1/2) = 1/12$. Hence, the volume of S is exactly one-half the volume of P.

The next extension of this problem (cutting a segment into five parts) would seem to indicate an extension into a four-dimensional coordinate system. But let us go back. Perhaps we can guess the answer. We consider the probability of failure in each case. When we divided the segments into three parts, the failure set F consisted of the three corners of the triangle (Figure 4), each having an area one-fourth the area of S. Thus the probability of failure, $\Pr(F) = 3(1/4) = 3(1/2)^2$.

When we divided the line segment into four parts, the failure set F consisted of the four corners of set P (Figure 6), each having a volume one-eighth the volume of P. Thus $\Pr(F) = 4(1/8) = 4(1/2)^3$. Although we don't have much data from which to extrapolate, we expect (hope) that if we partition the line into five parts, $\Pr(F) = 5(1/2)^4$, and if we partition into n parts $\Pr(F) = n(1/2)^{n-1}$.

Such a simple formula, if correct, would seem to suggest that we have overlooked a simple proof. We have. Suppose our unit segment has been partitioned into n parts by $n - 1$ points whose coordinates are $x_1, x_2, \ldots, x_{n-1}$ (Figure 7).

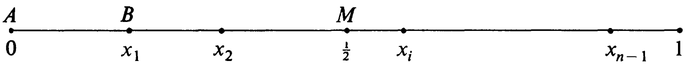

Figure 7.

Construction of an n-gon will be impossible if AB is longer than $1/2$, i.e., if $x_1 > 1/2$. For this to happen, all $n - 1$ points must fall to the right of the midpoint. The probability that $x_1 > 1/2$ equals one-half. The same is true for each of the other points. The probability that all $n - 1$ points lie to the right of the midpoint is $(1/2)^{n-1}$. Thus the probability that one of the n segments is too long is $(1/2)^{n-1}$. The same holds true for each of the n segments. Thus the probability that one of the n segments is too long is $n(1/2)^{n-1}$, and the probability of success is $1 - n(1/2)^{n-1}$.

The next problem, Buffon's Needle Problem, is a well-known problem of its kind, and it illustrates the power of this technique to make the solution of a seemingly difficult problem quite simple.

Problem 5. *Buffon's Needle Problem.* A plane is lined with parallel lines 2″ apart. A needle 2″ long is tossed onto the plane. What is the probability that the needle crosses a line?

Solution. Rather than consider the entire plane, we will concentrate on the points 1″ above the given line. Think of the needle as an arrow two inches long. Its position can be described by the ordered pair (θ, y), where y is the distance of its center from the line, and θ is the angle the arrow makes with the parallel lines. (See Figure 8.)

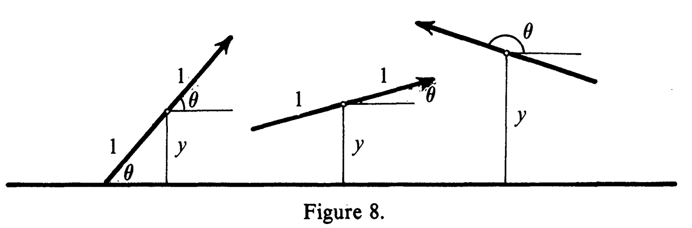

Figure 8.

Without loss of generality we restrict y and θ so that our sample space is defined as follows:

$$U = \begin{cases} 0 \leqslant y \leqslant 1 \\ 0 \leqslant \theta \leqslant \pi \end{cases}$$

From Figure 8 we can conclude that the arrow will cross the line if $y \leqslant \sin\theta$. The set of points in U that represent line crossings we will call set S.

$$S = \begin{cases} 0 \leqslant \theta \leqslant \pi \\ y \leqslant \sin\theta \end{cases}$$

Both sets are graphed in Figure 9.

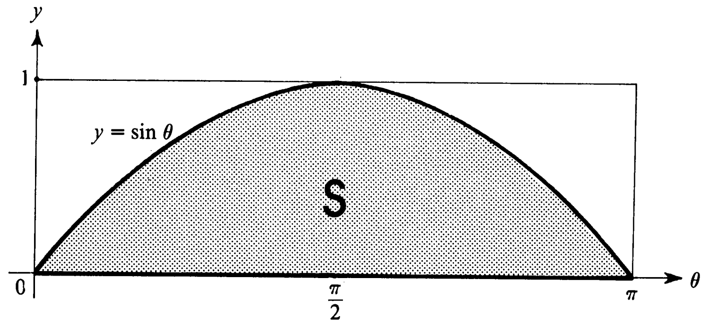

Figure 9.

Only those points in the shaded area represent arrows which will cross the line. Thus the area of S divided by the area of U is the required probability.

$$\frac{\text{Area } S}{\text{Area } U} = \frac{\int_0^{\pi} \sin\theta \, d\theta}{\pi} = \frac{2}{\pi}.$$

Our answer, surprisingly, involves the number pi. By repeated tossings of the needle onto the parallel lines, we may get an approximation of pi.

Problem 6. Suppose the needle in the previous problem is bent at its midpoint at an angle of 90 degrees and then tossed onto the plane of parallel lines. What is the probability that the bent needle will intersect a line?

Solution. Using the same dimensions as the previous problem, we will again consider only the points 1″ above a given line. The position of the bent needle can again be defined in terms of y and θ. See the examples in Figure 10.

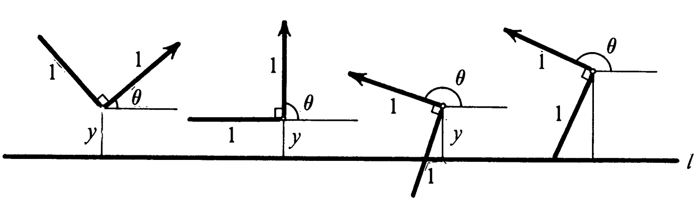

Figure 10.

Assuming that a random toss means that any angle θ, $0 \leqslant \theta \leqslant 2\pi$, and any distance y, $0 \leqslant y \leqslant 1$, is equiprobable, we picture the solution below. (Figure 11.)

$$U = \begin{cases} 0 \leqslant \theta \leqslant 2\pi \\ 0 \leqslant y \leqslant 1 \end{cases} \qquad S = \begin{cases} 0 \leqslant \theta \leqslant 2\pi \\ 0 \leqslant y \leqslant 1 \\ y \leqslant -\cos\theta, \quad \pi/2 \leqslant \theta \leqslant 5\pi/4 \\ y \leqslant -\sin\theta, \quad 5\pi/4 \leqslant \theta \leqslant 2 \end{cases}$$

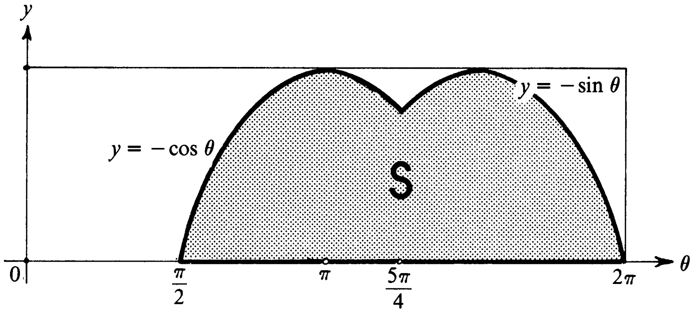

Figure 11.

Probability of intersection, Pr(S), equals (area S)/(area U).

$$\text{Area } S = \int_{\pi/2}^{5\pi/4} -\cos\theta\, d\theta + \int_{5\pi/4}^{2\pi} -\sin\theta\, d\theta = 2 + \sqrt{2},$$

$$\text{Area } U = 2\pi,$$

$$\Pr(S) = \frac{2+\sqrt{2}}{2\pi}.$$

Problem 7. From all triangles whose longest side is one unit, one triangle is selected at random. What is the probability that the triangle selected is obtuse?

We will present here three different solutions which, surprisingly, give three different answers. By the first method, we let x and y represent the lengths of the other two sides. It is clear that $0 < x < 1$, $0 < y < 1$, and $x + y > 1$. Triangle XYZ in Figure 12 represents our sample space. To be obtuse the additional condition is $x^2 + y^2 < 1$, which is represented by the shaded area. The probability that the

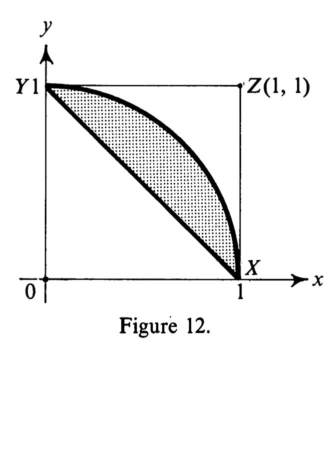

Figure 12.

triangle is obtuse is

$$\left(\frac{\pi}{4}-\frac{1}{2}\right)/\frac{1}{2}\approx .5708.$$

Using the second method, we consider the segment AB to be the unit segment (Figure 13). Considering only those points which lie above the line AB, vertex C must lie within the figure bounded by the two 60 degree arcs AC and BC. If the

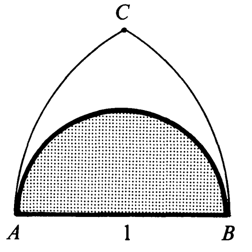

Figure 13.

random triangle is selected by choosing vertex C at random, the triangle will be obtuse if vertex C lies within the semicircle. The probability that the triangle will be obtuse is the ratio of the area of the semicircle to the area of the entire figure. The probability that the triangle is obtuse equals

$$\frac{\pi}{8}/\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)\approx .6394.$$

For the third method we use the figure of method two, but let vertex C be represented by polar coordinates (r,θ). (Figure 14.) If the random triangle is selected by choosing r and θ at random, an acceptable triangle is formed if $0<r<1$, $0<\theta<\pi/2$, and $r<2\cos\theta$. If the triangle is to be obtuse, we must add the condition $r<\cos\theta$. The shaded area equals $\int_0^{\pi/2}\cos\theta\,d\theta=1$ and the total area is

$$\frac{\pi}{3}+\int_{\pi/3}^{\pi/2}2\cos\theta\,d\theta=\frac{\pi}{3}+2-\sqrt{3}.$$

Thus the probability that the triangle is obtuse is

$$1/\left(\frac{\pi}{3}+2-\sqrt{3}\right)\approx .7604.$$

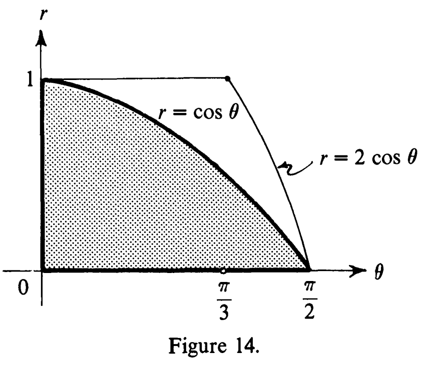

Figure 14.

Each of these answers is equally valid. The cause of this paradox lies in the imprecise statement of the problem. The method of selection of the random triangle must be stated or implied in the statement of the problem since, as we have seen, different methods result in different conclusions. Other methods are possible. Suppose, for example, the unit segment is taken as the base of the triangle and the two base angles are chosen at random. What is the probability that the triangle formed is obtuse? We leave this as a simple exercise for the reader.

EXERCISES

1. Two numbers are chosen at random between 0 and 1. What is the probability that the sum of their squares is greater than 1?
2. Three numbers are chosen at random between 0 and 10. Find the probability that the sum of their squares is less than 50.
3. Suppose a 2″ piece of wire is bent into the form of a circle and tossed onto a plane ruled with parallel lines 2″ apart. What is the probability that the circle touches a line?
4. A chord is picked at random from a circle. What is the probability the chord is longer than the radius?
5. In the figure below two points, A and B, are selected at random on segments WX and YZ. If $WX = ZY = 2$, and if $XM = MY = 1$, find the probability that triangle BAM is obtuse.

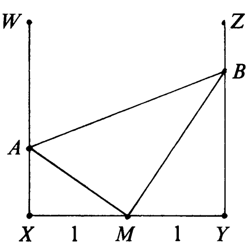

Figure 15.

6. Triangle RST is an equilateral triangle with M the midpoint of RS. Points A and B are selected at random, one on side RT and the other on side ST. Find the probability that triangle AMB is obtuse.

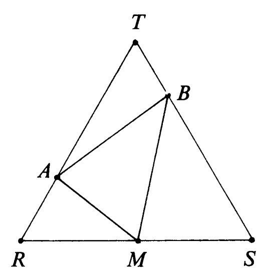

Figure 16.

7. Three points are chosen at random one on each side of an equilateral triangle. What is the probability that the triangle formed by these points is obtuse?