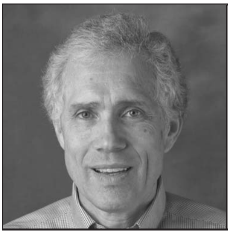
Designing a Table Both Swinging and Stable

Greg N. Frederickson



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Some mathematical ideas are just so nifty that people can't resist finding a physical realization for them. One example is a hinged dissection that transforms an equilateral triangle to a square, used by the geometer and math historian Howard Eves to make a reconfigurable table. However, difficulties can arise when we step from the mathematical world into the physical world. In this article we propose a surprising new solution to handle those difficulties.

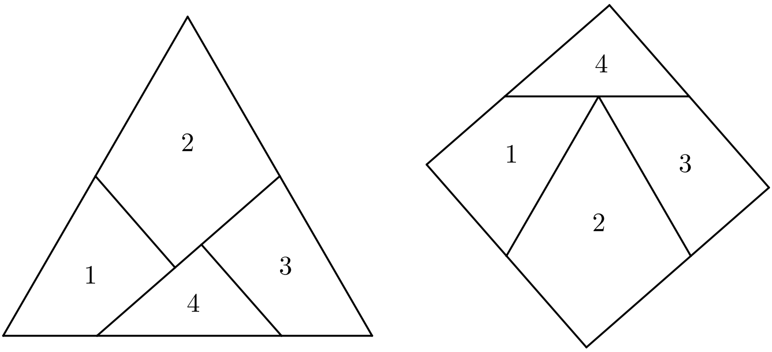

Figure 1. Hingeable dissection of a triangle to a square.

A geometric dissection is a cutting of a geometric figure into pieces that can be rearranged to form another figure [**8**], [**12**]. Such visual demonstrations of the equivalence of area span recorded history from the geometric explorations of the ancient Greeks (see [**1**] and [**3**]) to recent listings on the World Wide Web [**13**]. One of the most remarkable geometric dissections is the 4-piece dissection of an equilateral triangle to a square shown in Figure 1, first discovered in 1902 by either Henry E. Dudeney or Charles W. McElroy [**4**]. Dudeney introduced an intriguing variation in his book *The Canterbury Puzzles* [**5**]. After presenting the remarkable 4-piece solution, he wrote,

> I add an illustration showing the puzzle in a rather curious practical form, as it was made in polished mahogany with brass hinges for use by certain audiences. It will be seen that the four pieces form a sort of chain, and that when they are closed up in one direction they form the triangle, and when closed in the other direction they form the square.

This hinged model (shown in Figure 2) has captivated readers ever since. It is just too nifty not to have been described in at least a dozen other books in the intervening years. Even with the recent publication of a book wholly devoted to the subject of hinged dissections [**9**], this model remains the ultimate crowd-pleaser.

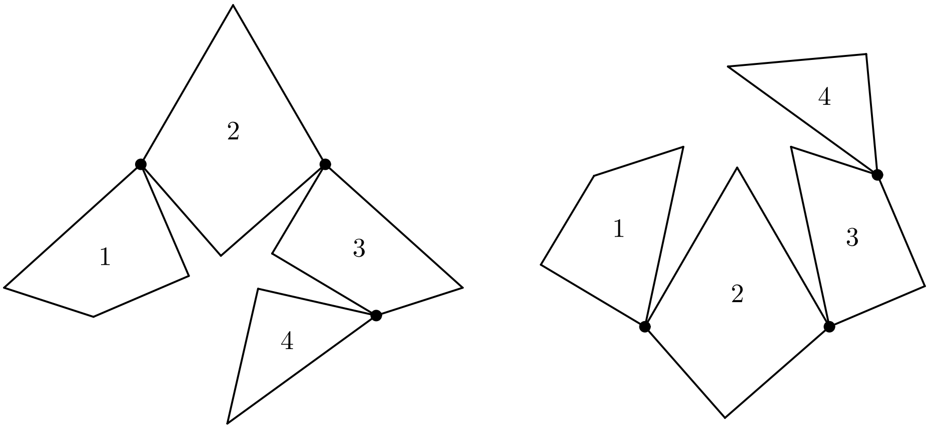

Figure 2. Hinged dissection of a triangle to a square.

It should not be a surprise that this dissection has served as a source for mathematically motivated design. Howard Eves [**7**, pp. 37–38] described a set of four connected tables that he had built that would swing around to form either a square or a triangular top. As Eves recounted [**6**],

> I made a table; it's the size of a card table. It's meant for playing games upon. And it's in the form of, say, a square, like most card tables are, and you might be playing cards and somebody decides to withdraw so you only need three players. So just unhooking a few little hooks, you swing the thing around, and behold, it goes into an equilateral triangular table. It makes a conversation piece, if nothing else. You might think that it needs a great proliferation of legs, but it doesn't. Four legs suffice, and they should be placed not at the corners but roughly at the centroids of the four pieces, so that the legs are inside but they are still far enough apart to give stability to the table.

Eves was not the only one to have built a table based on this dissection. Two other attempts have different schemes for placing legs on the pieces. A coffee table designed by the Israeli artist Maty Grünberg and produced by London art gallery owner Zeev Aram had a leg at each corner of each piece, for a total of 15 legs [**2**]. A full-height table built by the craftsman Jan de Koning for Joop Van Der Vaart had a leg at each corner of the equilateral triangle and at each corner of the square, for a total of 7 legs [**10**].

Eves's scheme has the fewest legs of the three schemes. Yet with one leg positioned at the centroid of each piece, the legs are fairly close to the center of both the triangle and the square, and this contributes to instability. If the table is left in an open position (with the four individual tables strung out), then having so few legs also causes problems, since the three hinges will be subjected to various torques. On the other hand, the more legs that are used, the harder it is to get them all to touch the floor simultaneously when the table is closed up in either configuration. If the floor is at all uneven, then the chances of getting all the legs to touch the floor decreases quickly as the number of legs increases.

Thus each of the schemes seems to have problems with stability. Is there some hinged dissection of an equilateral triangle to a square that is free of such difficulties? This paper addresses that question by using a familiar technique to create an unexpected new solution.

Crossposing strips

Before examining the new solution, let's review how the *strip technique* produces the triangle-to-square dissection in Figure 1 [**8**], [**12**]. We begin with a pair of infinite strips, one of equilateral triangles and the other of squares of the same area as the triangles. Then, as shown in Figure 3, we *crosspose* the strip of squares over the strip of equilateral triangles, so that the common area (in the overlap) is precisely twice the area of each polygon. We perform the crossposition so that each point of two-fold rotational symmetry in the overlap is either crossed by a boundary of the other strip or covered by a point of two-fold rotational symmetry in the other strip. The line segments in one strip induce cuts in the other strip, giving the dissection in Figure 1. The angle between the two strips in Figure 3 turns out to be $\arcsin(\sqrt[4]{3}/2)$, or about $41.1503°$. Once we know this angle, we can readily determine the angles of every piece.

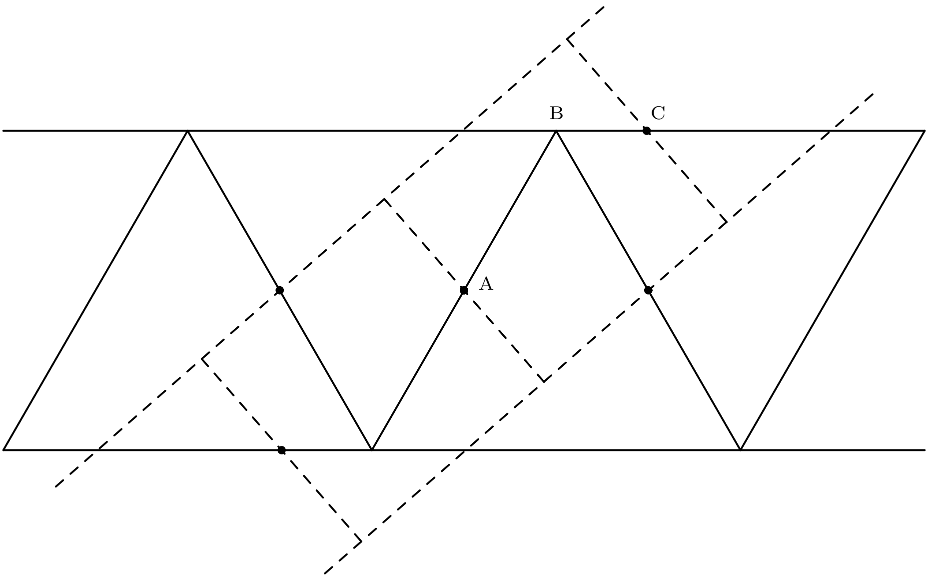

Figure 3. Crossposition of triangles and squares.

The crossposition leads to a hinged dissection because it induces an associated tessellation of triangles and another tessellation of squares. Every crossing of a line segment in one tessellation with a line segment in the other turns out to be a point of rotational symmetry. These points of symmetry originate from three points of rotational symmetry within each strip (the small dots in Figure 3) and from points where the boundaries of the two strips intersect. Any two pieces that share a point of rotational symmetry can then be hinged together at that point.

Analysis of the original solution

Rather than support each piece with legs, we could place a leg at each corner of the largest piece in the dissection, P_2. However, this leaves much of the table top supported only through the hinges, thus exerting considerable torque on those hinges.

To quantify this problem, we calculate the area of each piece. Assume that the square has sides of length 1, and the triangle has sides of length s. By equating the two areas, we find that $s = 2/\sqrt[4]{3}$. We first compute the area of piece P_4, a right triangle with legs of length .5 and $(s/4)\sqrt{4-\sqrt{3}}$.

$$\text{Area}(P_4) = \frac{1}{24}\sqrt{-9+12\sqrt{3}} \approx .1430$$

To compute the area of P_2, we view it as a copy of P_4 attached to a small equilateral triangle of area one quarter of the full equilateral triangle.

$$\text{Area}(P_2) = \frac{1}{24}\left(6+\sqrt{-9+12\sqrt{3}}\right) \approx .3930$$

To compute the area of P_3, we apply the law of cosines to analyze triangle ABC.

$$\text{Length(BC)} = \frac{1}{6}\left(-\sqrt{3\sqrt{3}}+3\sqrt{4-\sqrt{3}}\right) \approx .3731$$

$$\text{Area(ABC)} = \frac{1}{8}\left(\sqrt{4\sqrt{3}-3}-1\right) \approx .1227$$

The combined area of P_2 and P_3 is the area of triangle ABC plus one half the area of the square. Subtraction gives us first P_3 and then P_1.

$$\text{Area}(P_3) = \frac{1}{24}\left(3+\sqrt{-108+66\sqrt{3}}\right) \approx .2297$$

$$\text{Area}(P_1) = \frac{1}{24}\left(15-\sqrt{36+30\sqrt{3}}\right) \approx .2342$$

From our analysis, we see that P_2, with area about .3930, is the largest piece. However, with less than half the total area, P_2 is a poor choice to support the other three pieces.

A more stable table

Like many lovely things, the improved solution to the table problem was discovered by accident. I was playing around with the elements of Figure 3 and wondered what would happen if I crossposed the two strips in the way shown in Figure 4. The good news is that this crossposition produces a very large piece. The bad news is that there are now six pieces, not four. We shall use N_i to refer to the ith piece in the new dissection.

Actually, there is more bad news. The resulting dissection, shown in Figure 5, is not hingeable. There is no way to put a hinge on N_5 so that it can swing from the top of the equilateral triangle to the bottom of the square. This is a near-miss, as all of the other

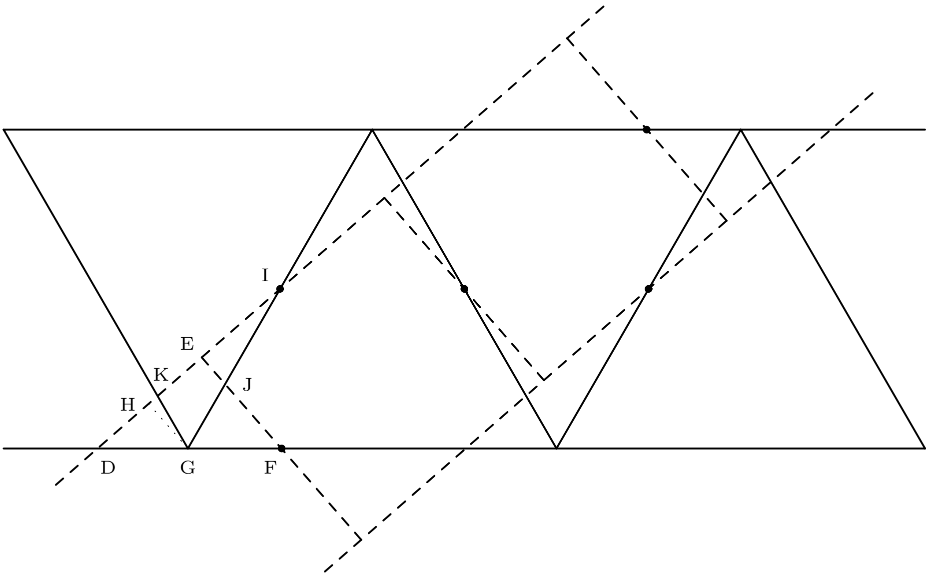

Figure 4. New crossposition for a triangle to a square.

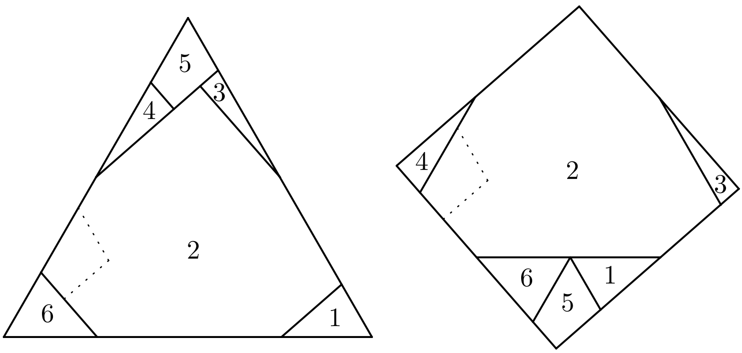

Figure 5. New not-completely-hingeable triangle to a square.

pieces can be hinged together. Yet there is a curious fact: If we cut a piece congruent to N_5 out of N_2, as indicated by the dotted lines, we can swing N_5 from the top of the equilateral triangle into the resulting cavity. We can also swing N_5 from the bottom of the square into that cavity.

This suggests our solution: Cut out a seventh piece N_7 and hinge it to N_6 in the equilateral triangle, as we hinge N_5 to N_4 in the equilateral triangle. We can then use N_4 to swing N_5, and N_6 to swing N_7, into the right positions. We see the new hingeable dissection in Figure 6 and the appropriate hinging in Figure 7. Since N_2 in Figure 6 contains well over half of the area, this design begs for just one leg, positioned near the center of that piece. The table would then be a *pedestal table*.

Note that the intermediate configurations in Figure 7 are meant to be descriptive rather than to convey a specific sequence of movements. To avoid pieces binding against one another, keep N_7 flush against N_6 as you initially rotate N_6 from its position in the square. Also keep N_5 flush against N_4 as you rotate N_4 from its position in the square.

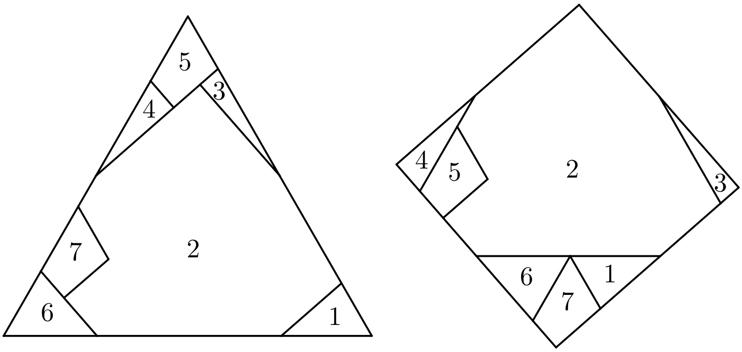

Figure 6. New hingeable triangle to a square.

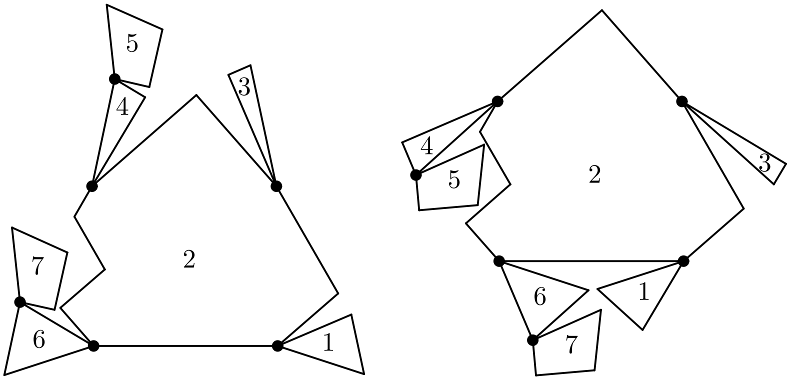

Figure 7. New hinged triangle to a square.

Analysis of our new solution

In terms of stability, how good is our new solution? We determine the area of N_2 as follows. Consider N_3, N_4, and N_5 that form a "cap" in the equilateral triangle. Note that the cap extends down to the midpoints of the two sides of the equilateral triangle. Also consider N_1, N_6, and N_7 that form the bottom corner of the square. The bottom corner of the square can nestle exactly into the cap, forming an equilateral triangle whose side length is precisely one half that of the full equilateral triangle. Thus the combined areas of N_1, N_3, N_4, N_5, N_6, and N_7 is .25, leaving the rest for N_2.

$$\text{Area}(N_2) = .75$$

To determine the sizes of the remaining pieces, we refer to Figure 4. We already know the lengths of DF, EF, and DE. Let GH be the altitude to side DI of triangle DGI. We can then determine lengths of line segments DG, GH, DH, HI, EI, and EJ, using simple techniques such as similar triangles. Each of the above lengths can be derived in closed form, using a system such as Mathematica. We then compute the areas of the other pieces.

$$\text{Area}(N_4) = \frac{1}{48}\left(-51 - 57\sqrt{3} + \sqrt{10908 + 6906\sqrt{3}}\right) \approx .0312$$

$$\text{Area}(N_6) = \frac{1}{16}\left(-15 - 19\sqrt{3} + 9\sqrt{12 + 10\sqrt{3}}\right) \approx .0515$$

$$\text{Area}(N_1) = \frac{1}{16}\left(-7 - \sqrt{3} + \sqrt{36 + 30\sqrt{3}}\right) \approx .0404$$

$$\text{Area}(N_5) = \frac{1}{24}\left(33 + 30\sqrt{3} - 2\sqrt{828 + 534\sqrt{3}}\right) \approx .0511$$

$$\text{Area}(N_3) = \frac{1}{48}\left(-3 - 3\sqrt{3} + \sqrt{36 + 30\sqrt{3}}\right) \approx .0246$$

We now return to Figure 5, before N_7 was cut out of what was initially N_2. Adding the areas of N_7 and the resulting N_2 gives

$$\frac{1}{24}\left(51 + 30\sqrt{3} - 2\sqrt{828 + 534\sqrt{3}}\right) \approx .8011$$

as the area of the largest piece of that partially-hingeable dissection.

It can be argued that the size of the largest possible piece for any dissection is approximately .8254, corresponding to the common overlap when a side of the square rests in the center of a side of the equilateral triangle. Thus N_2 in the intermediate Figure 5 is very close to the largest possible size for a piece in any triangle-to-square dissection, hinged or not. We conclude that our table will have great stability even if N_2 is the only piece supported either by several legs or by a single pedestal leg.

Positioning the pedestal

Where should we position the pedestal? For a table that is not reconfigurable, we would obviously position it under the center of gravity of the top. With two different configurations, one solution is to position the pedestal midway between the centers of gravity of the square and the equilateral triangle. Serendipitously, these two centers of gravity are relatively close to each other in our new table, as we now show.

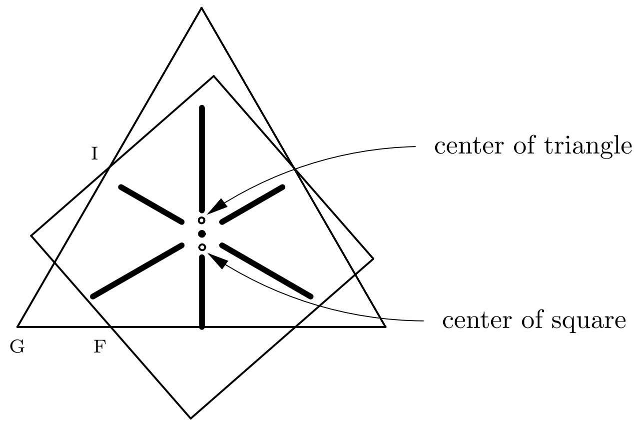

Figure 8. Centers of gravity and pedestal struts.

In Figure 8 we see the equilateral triangle and the square superimposed on N_2. The center of gravity of the equilateral triangle is directly above the midpoint of the base, one third of the way from that midpoint to the apex. That makes its elevation above the base one third of the height of the triangle.

As for the square, its center of gravity is halfway between vertex F and the midpoint of the opposite side, which is also the midpoint of the right side of the equilateral triangle. Thus the center of gravity of the square is above the base of the equilateral triangle by a distance that is a quarter of the height of that triangle.

The center of gravity of the triangle is

$$\frac{s}{4} - \mathrm{BC} = \frac{1}{6}\left(2\sqrt{3\sqrt{3}} - 3\sqrt{4-\sqrt{3}}\right) \approx .00685$$

to the right and $(s/24)\sqrt{3} = \sqrt[4]{3}/12 \approx .1097$ above the center of gravity of the square. Thus they are approximately .1099 apart, which means that any reasonably broad pedestal positioned midway between the two centers of gravity will do just fine.

The point midway between the two centers of gravity is marked by the small black dot. If we position a pedestal leg at this point, and use the six struts indicated by thickened line segments, there should be good support for the top, whether it is configured as the equilateral triangle or the square.

Finishing touches

Howard Eves mentioned "little hooks," which he undoubtedly used for locking the pieces together when his table was in either form. Although he didn't say where the hooks should be positioned, it is clear that one hook would fasten P_1 and P_4 together at the straight angles formed by their shared corners. To control warping of the top and to increase rigidity, Eves might also have used additional hooks to lock the corners of the triangle together in the square, and similarly to lock the corners of the square together in the triangle. Of course, good craftsmanship dictates that the hooks be fastened on the underside of the table.

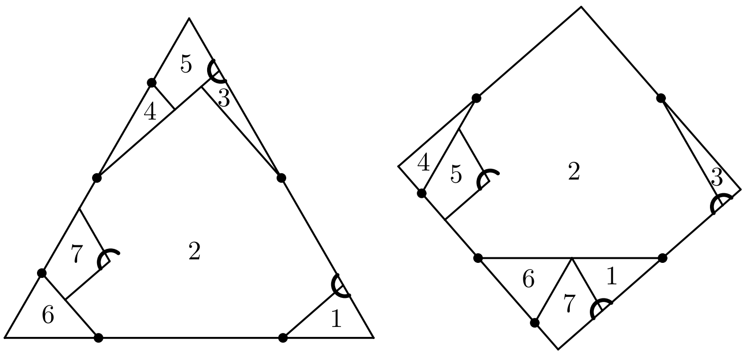

Figure 9. Position of hooks for the new hinged table.

Hooks would also help our new table as well, as we can see in Figure 9. Let's use three hooks, indicated by the half-circles. For easy reference, dots mark the positions of the six hinges. Place one hook on N_3 so that it locks with N_5 in the triangle and with N_2 in the square. Place a second hook on N_1 so that it locks with N_2 in the triangle and with N_7 in the square. Place the third hook on N_2 so that it locks with N_7 in the triangle and with N_5 in the square. Since six of the pieces are relatively small, there should be a much smaller risk of warping or lack of rigidity, so we probably do not need more than the three hooks. Readers can enjoy animations of Figures 2, 7, and 9 that are posted on a webpage [**11**].

Besides hingeability, we should note one other neat feature of the dissections in this article. Both dissections have a lovely “grain-preserving” property [**9**]. If the pieces are cut from wood with a nice parallel grain that is aligned in one direction in the equilateral triangle, then the grain will be nicely aligned in the square too. This follows immediately from the fact that each piece rotates precisely 180° on its hinge, relative to the other piece on the hinge. If we use wood with a straight, clear grain for our table top, then both the natural and the mathematical beauty of our hinged table can only be enhanced by lining up the grain in both configurations. What a wonderful bonus to complement the increased stability of our new table!

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Eves’s Original Hinged Table