CHAPTER 1

Basics

1.1 Sets

Set theory is the bedrock of all of modern mathematics. A *set* is a collection of objects. We usually denote a set by an upper case roman letter. If S is a set and s is one of the objects in that set then we say that s is an *element of* S and we write $s \in S$. If t is not an element of S then we write $t \notin S$.

Some of the sets that we study will be specified just by listing their elements: $S = \{2, 4, 6, 8\}$. More often we shall use *set-builder notation*: $S = \{x \in \mathbb{R} : 4 < x^2 + 3 < 9\}$. This last is read "the set of x in the reals such that $x^2 + 3$ lies between 4 and 9."

The collection of all objects not in the set S is called the *complement* of S and is denoted by ${}^cS$. The complement of S must be understood in the context of some "universal set"—see Example 1.1.

If S and T are sets and if each element of S is also an element of T then we say that S *is a subset of* T and we write $S \subseteq T$. If S is not a subset of T then we write $S \not\subseteq T$.

EXAMPLE 1.1.1. Let

$$S = \{a, b, c, d, e\}, \quad T = \{a, c, e, g, i\}, \quad \text{and} \quad U = \{c, d\}.$$

Then

$$a \in S, \;\; a \in T, \;\; d \in S, \;\; d \notin T, \;\; U \subseteq S, \;\; U \not\subseteq T.$$

If the universe is understood to be the standard 26-letter roman alphabet, then it follows that

$${}^cT = \{b, d, f, h, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}. \qquad \square$$

1.2 Operations on Sets

If S and T are sets then we let $S \cap T$ denote the collection of all objects that are both in S and in T. We call $S \cap T$ the *intersection* of S and T.

In case $S_1, S_2, S_3, \ldots$ are sets then the collection of all objects common to all the S_j, called the *intersection* of the S_j, is denoted by

$$\bigcap_{j=1}^{\infty} S_j \qquad \text{or} \qquad \bigcap_j S_j \, .$$

If S and T are sets then we let $S \cup T$ denote the collection of all objects that are either in S or in T or both. We call $S \cup T$ the *union* of S and T.

In case $S_1, S_2, S_3, \ldots$ are sets then the collection of all objects that lie in at least one of the S_j, called the *union* of the S_j, is denoted by

$$\bigcup_{j=1}^{\infty} S_j \qquad \text{or} \qquad \bigcup_j S_j \, .$$

Figure 1.1 illustrates the concepts of intersection and union, by way of what is known as a *Venn diagram.*

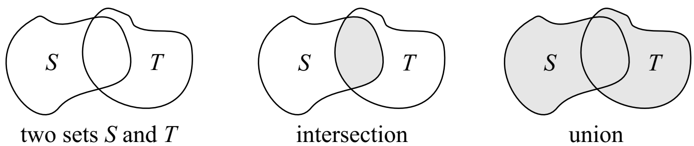


FIGURE 1.1. Venn diagram of an intersection and a union.

EXAMPLE 1.2.1. Let $S = \{1, 2, 3, 4, 5\}$ and $T = \{2, 4, 6, 8, 10\}$. Then

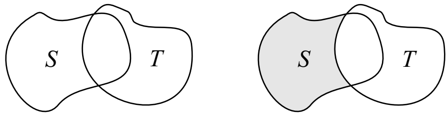
$$S \cap T = \{2, 4\} \qquad \text{and} \qquad S \cup T = \{1, 2, 3, 4, 5, 6, 8, 10\} \, . \qquad \square$$

If S and T are sets then we let

$$S \times T \equiv \{(s, t) : s \in S \text{ and } t \in T\} \, .$$

We call $S \times T$ the *cartesian product* of S and T. Observe that $S \times T$ and $T \times S$ are distinct. Sometimes we will take the product of finitely many sets $S_1, S_2, \ldots, S_k$. Thus

$$S_1 \times S_2 \times \cdots \times S_k = \{(s_1, s_2, \ldots, s_k) : s_j \in S_j \text{ for all } j = 1, \ldots, k\} \, .$$



two sets S and T $\qquad$ $S \setminus T$

FIGURE 1.2. Venn diagram of a set-theoretic difference.

If S and T are sets then we let their *set-theoretic difference* be

$$S \setminus T \equiv \{x \in S : x \notin T\} .$$

If $S, T \subseteq \mathbb{R}$, the real numbers, then ${}^c S = \mathbb{R} \setminus S$ and $S \setminus T = S \cap {}^c T$. Figure 1.2 illustrates the concept of set-theoretic difference.

EXAMPLE 1.2.2. Let $S = \{a, b, 1, 2\}$, $T = \{b, c, d, 2, 5\}$, and $U = \{\alpha, \beta\}$. Then

$$S \setminus T = \{a, 1\} \qquad \text{and} \qquad T \setminus S = \{c, d, 5\} .$$

Also

$$S \times U = \{(a, \alpha), (b, \alpha), (1, \alpha), (2, \alpha), (a, \beta), (b, \beta), (1, \beta), (2, \beta)\}$$

and

$$U \times S = \{(\alpha, a), (\alpha, b), (\alpha, 1), (\alpha, 2), (\beta, a), (\beta, b), (\beta, 1), (\beta, 2)\}. \qquad \square$$

We conclude by noting that there is a distinguished set that will arise frequently in our work. That is the *empty set* $\emptyset$. The empty set is the set with no elements. Observe that $\emptyset \subseteq A$ for any set A.

1.3 FUNCTIONS

Let S and T be sets. A *function* f from S to T is a rule that assigns to each element of S a unique element of T.

EXAMPLE 1.3.1. Let $S = \{1, 2, 3\}$ and $T = \{a, b\}$. The rule

$$\begin{aligned} 1 &\longrightarrow a \\ 2 &\longrightarrow a \\ 3 &\longrightarrow b \end{aligned}$$

is a function, because it assigns a unique element of T to each element of S. It assigns the same element of T to each of 1 and 2 in S; that is allowed.

$\square$

We write $f : S \to T$ if f is a function from S to T. We call S the *domain* of f and we call T the *range* of f.

EXAMPLE 1.3.2. Let $S = \{a, b, x\}$ and $T = \{1, \alpha, \gamma\}$. Define the function f by

$$f : \begin{cases} a \to \alpha \\ b \to 1 \\ x \to \alpha \end{cases}$$

Then the domain of f is the set S itself. There are several choices for the range. The set $\{\alpha, 1\}$ can be said to be the range. Also the entire set T can be said to be the range. □

Many of our functions are given by formulas. If we write, for example, $f(x) = \sqrt{2x + 5}$, then we mean that the function f assigns to each number x the number obtained by doubling x and adding 5 and then taking the square root. We understand the domain of f to be all numbers x for which the formula defining f makes sense—for this example, the domain is $\{x : x \geq -5/2\}$. We understand the range of f to be any set containing all the values of f—for this example, the range could be taken to be $\{y : y \geq 0\}$.

If the function f has domain S and range T, and if for each element $t \in T$ there is some $s \in S$ such that $f(s) = t$ then we say that f is *onto*.

If the function g has domain S and range T, and if the only way that $f(s_1)$ can equal $f(s_2)$ is if $s_1 = s_2$ then we say that f is *one-to-one*.

EXAMPLE 1.3.3. Let $S = \{-3, -2, -1, 0, 1, 2, 3\}$ and let $T = \{0, 1, 4, 9\}$. Let the function f be given by $f(x) = x^2$. Then the set of all values of f, applied to elements of S, is $\{0, 1, 4, 9\}$. Therefore f is onto. However notice that $f(-2) = f(2) = 4$. Therefore the function f is not one-to-one. □

1.4 OPERATIONS ON FUNCTIONS

Let f and g be functions with domain S and range T. We define

- $[f + g](x) \equiv f(x) + g(x)$
- $[f - g](x) \equiv f(x) - g(x)$
- $[f \cdot g](x) \equiv f(x) \cdot g(x)$
- $\left[\frac{f}{g}\right](x) \equiv \frac{f(x)}{g(x)}$ provided that $g(x) \neq 0$

EXAMPLE 1.4.1. Let $f(x) = x^3 - x$ and $g(x) = x^4$. Then

$$[f + g](x) = x^3 - x + x^4\,, \qquad [f - g](x) = x^3 - x - x^4\,,$$

$$[f \cdot g](x) = (x^3 - x) \cdot x^4 = x^7 - x^5\,, \qquad \left[\frac{f}{g}\right](x) = \frac{x^3 - x}{x^4}. \qquad \square$$

If $f : S \to T$ and $g : T \to U$ then we may consider the function $g \circ f$ defined by

$$(g \circ f)(x) = g(f(x))\,.$$

If $f : S \to T$ is both one-to-one and onto then we may define a function f^{-1} by the rule $f^{-1}(x) = y$ if and only if $f(y) = x$. We call f^{-1} the *inverse* of the function f. We have the essential properties

$$(f \circ f^{-1})(t) = t \quad \forall t \in T$$

and

$$(f^{-1} \circ f)(s) = s \quad \forall s \in S\,.$$

EXAMPLE 1.4.2. Let $f(x) = x^2 - 3x$ and $g(x) = x^3 + 1$. Then

$$(f \circ g)(x) = (x^3 + 1)^2 - 3 \cdot (x^3 + 1)$$

and

$$(g \circ f)(x) = (x^2 - 3x)^3 + 1\,.$$

The function f is not one-to-one because $f(0) = f(3) = 0$. But $g : \mathbb{R} \to \mathbb{R}$ is both one-to-one and onto. We may solve the equation

$$(g \circ g^{-1})(x) = x$$

to find that

$$[g^{-1}(x)]^3 + 1 = x$$

or

$$g^{-1}(x) = \sqrt[3]{x - 1}\,. \qquad \square$$

1.5 NUMBER SYSTEMS

The most rudimentary number system is the *natural numbers*. These are the counting numbers $1, 2, 3, \ldots$, which are denoted by the symbol $\mathbb{N}$. Of the four standard arithmetic operations, the natural numbers are closed only under addition and multiplication.

The *integers* comprise both the positive and negative whole numbers and also 0. We denote this set by $\mathbb{Z}$. Of the four standard arithmetic operations, the integers are closed under addition, subtraction, and multiplication.

The *rational numbers* consist of all quotients of integers. Thus m/n is a rational number provided that $m, n \in \mathbb{Z}$ and $n \neq 0$. We denote the set of all rational numbers by $\mathbb{Q}$. The set of rational numbers is closed under all four of the standard arithmetic operations, except that division by 0 is not allowed.

EXAMPLE 1.5.1. The number 4 is a natural number. Of course it is also an integer. Writing it as $4 = 4/1$ we also see that this number is a rational number.

The number -6 is an integer. It is not a natural number. Writing it as $-6 = (-6)/1$, we also see that this number is a rational number.

The number $2/3$ is neither an integer nor a natural number. But it is a rational number. □

The number system of greatest interest to us is the real number system. It contains the rational numbers, and has several other interesting properties as well. We explore the real numbers in the next subsection.

1.5.1 THE REAL NUMBERS

The rational numbers are a *field*. This means that there are operations of addition ($+$) and multiplication ($\times$) that satisfy the usual laws of arithmetic. In addition the field $\mathbb{Q}$ satisfies certain properties of the ordering ($<$):

1. If $x, y, z \in \mathbb{Q}$ and $y < z$ then $x + y < x + z$.

2. If $x, y \in \mathbb{Q}$, $x > 0$, and $y > 0$ then $x \cdot y > 0$.

Thus $\mathbb{Q}$ is an *ordered field*.

The real numbers will be an ordered field containing the rationals and satisfying an additional completeness property. We formulate that property in terms of *least upper bound*.

Definition 1.5.2. Let $S \subseteq \mathbb{R}$. The set S is called *bounded above* if there is an element $b \in \mathbb{R}$ such that $x \leq b$ for all $x \in S$. We call the element b an *upper bound* for the set S.

Definition 1.5.3. Let $S \subseteq \mathbb{R}$. An element $b \in \mathbb{R}$ is called a *least upper bound* (or *supremum*) for S if b is an upper bound for S and there is no upper bound b' for S that is less than b. We write $b = \sup S = \operatorname{lub} S$.

EXAMPLE 1.5.4. Let $S = \{x \in \mathbb{Q} : 3 < x < 5\}$. Then the number 9 is an upper bound for S, as is the number 7. The least upper bound for S is 5. We write $5 = \text{lub}\, S = \sup S$. □

By its very definition, if a least upper bound exists then it is unique.

Before we go on, let us record a companion notion for lower bounds:

Definition 1.5.5. Let $S \subseteq \mathbb{R}$. The set S is called *bounded below* if there is an element $c \in \mathbb{R}$ such that $x \geq c$ for all $x \in S$. We call the element c a *lower bound* for the set S.

Definition 1.5.6. Let $S \subseteq \mathbb{R}$. An element $c \in \mathbb{R}$ is called a *greatest lower bound* (or *infimum*) for S if c is a lower bound for S and there is no lower bound c' for S that is greater than c. We write $c = \inf S = \text{glb}\, S$.

By definition, if a greatest lower bound exists then it is unique.

EXAMPLE 1.5.7. Let $S = \{x \in \mathbb{R} : 0 < x < 1\}$ and $T = \{x \in \mathbb{R} : 0 \leq x < 1\}$. Then -1 is a lower bound both for S and for T and 0 is the greatest lower bound for both sets. We write $0 = \text{glb}\, S$ and $0 = \text{glb}\, T$. We may also write $0 = \inf S$ and $0 = \inf T$. Notice that $0 \notin S$ while $0 \in T$.

Also 5 is an upper bound both for S and for T, and 1 is the least upper bound for both sets. We write $1 = \text{lub}\, S$ and $1 = \text{lub}\, T$. We may also write $1 = \sup S$ and $1 = \sup T$. Observe that 1 is not in S and is not in T. □

Now we have:

Theorem 1.5.8. *There exists an ordered field* $\mathbb{R}$ *that* **(i)** *contains* $\mathbb{Q}$ *as a subfield and* **(ii)** *has the property that any non-empty subset of* $\mathbb{R}$ *that has an upper bound also has a least upper bound* (*that is also an element of* $\mathbb{R}$).

An equivalent, companion, statement is that if T is any set that is bounded below then T has a greatest lower bound (that is also an element of $\mathbb{R}$).

EXAMPLE 1.5.9. It is known (see [KRA2, page 114]) that there is no rational number whose square is 2—see Example 1.5.10 below. Let

$$S = \{x \in \mathbb{R} : x > 0 \text{ and } x^2 < 2\}.$$

Of course S is bounded above (by 2, for example), and so has least upper bound α. Of course α will be an element of $\mathbb{R}$, but $\alpha \notin \mathbb{Q}$. It can be shown that $\alpha^2 = 2$ (see [KRA1, Section 2.5, Theorem 12]). Thus the real number system contains numbers that are missing from the rational number system. These are called the *irrational numbers*. □

It can also be shown that the number π, which represents the ratio of the circumference of a circle to its diameter, is not a rational number. But π *does* exist as a real number.

EXAMPLE 1.5.10. Let us confirm that $\sqrt{2}$ is not a rational number. Suppose to the contrary that it is. So $\sqrt{2} = p/q$, with p and q integers. By division, we may suppose that p and q have no common divisors.

Thus

$$\left(\frac{p}{q}\right)^2 = 2\,.$$

Multiplying this out gives

$$2q^2 = p^2\,.$$

Since 2 divides the left side, we conclude that 2 divides the right side. So 2 divides p. Write $p = 2r$ for r an integer.

Thus we have

$$2q^2 = (2r)^2\,.$$

Simplifying gives

$$q^2 = 2r^2\,.$$

Since 2 divides the right side, we conclude that 2 divides the left side. So 2 divides q. We have shown that 2 divides p and also that 2 divides q. This contradicts the assumption that p and q have no common divisors.

We conclude that $\sqrt{2}$ cannot be rational. □

It is considerably more difficult to prove that π is irrational. We cannot treat the matter here, but see [NIV].

We shall learn below that the set of numbers $\mathbb{R} \setminus \mathbb{Q}$ (the irrational numbers) is much larger than $\mathbb{Q}$ itself. Thus "most" real numbers are not rational.

1.6 Countable and Uncountable Sets

Georg Cantor's theory of countable and uncountable sets, and more generally of many orders of infinity, is an integral part of any treatment of real analysis. What we give here is a summary. Complete treatments may be found in [KRA1, Section 1.8] and [KRA2, Section 5.8].

Two sets S and T are said to have the *same cardinality* if there is a one-to-one, onto function $\phi : S \to T$. We write $\operatorname{card} S = \operatorname{card} T$. In this context we refer to such a function ϕ as a *bijection*, or just an isomorphism.

The surprise is that some unlikely pairs of sets have the same cardinality. In particular, it is possible for $S \subseteq T$, $S \neq T$, and yet card S = card T.

EXAMPLE 1.6.1. Let $A = \{\heartsuit, \spadesuit, \clubsuit\}$, $B = \{\%, \&, \#\}$, and $C = \{1, 2\}$. Then

$$f: \quad \begin{cases} \heartsuit \to \# \\ \spadesuit \to \% \\ \clubsuit \to \& . \end{cases}$$

is a bijection of A to B. This gives mathematical confirmation of the obvious fact that A and B have the same cardinality. We write card A = card B.

On the other hand, it is impossible to construct a bijection from A to C. So A and C do not have the same cardinality. □

EXAMPLE 1.6.2. Let $S = \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$ (the even integers) and let $T = \mathbb{Z}$. Then obviously $S \subseteq T$ but $S \neq T$. Yet $\phi(n) = n/2$ is an isomorphism of S to T. So card S = card T. □

If two sets have the same cardinality, then we think of them as having the same size. For finite sets, this idea coincides with our intuition: two sets have the same cardinality if and only if they have the same (finite) number of elements. But for infinite sets this says something new.

If a set S has the same cardinality as $\mathbb{N}$, the natural numbers, then we say that S is *countable*.

EXAMPLE 1.6.3. Let $S = \mathbb{Z}$, the integers, and let $T = \mathbb{N}$, the natural numbers. Define

$$f(n) = (-1)^{n+1} \cdot \lfloor j/2 \rfloor .$$

Here $\lfloor \ \rfloor$ is the greatest integer function. Then f is a bijection of T to S. So S and T have the same cardinality. We say that the integers are a countable set. □

EXAMPLE 1.6.4. Let $S = \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$. The last two examples show that S is countable. A similar argument shows that $T = \{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$ is countable. We will see below that the set $\mathbb{R}$ of real numbers is *not* countable. We say that $\mathbb{R}$ is *uncountable*. □

Cantor's great insight was that the set $\mathbb{R}$ of real numbers is in fact not countable (see [KRA1, Section 1.8] or [KRA2, Subsection 5.8.3] for a proof). If S is infinite and has cardinality different from the cardinality of $\mathbb{N}$ then we say that S is *uncountable*.

We now list some of the key properties of countable sets:

1. If both S and T are countable then $S \cup T$, $S \cap T$, and $S \times T$ are at most countable.
2. If X is uncountable and $Y \supseteq X$ then Y is uncountable.
3. If X is countable and $Y \subseteq X$, then Y is at most countable.

The phrase "at most countable" means either countably infinite or finite or empty. It is common to refer to a set of this type as "denumerable" (although many sources do not make this distinction very clearly). We reserve the word "countable" for infinite sets that have the same cardinality as the set $\mathbb{N}$.

EXAMPLE 1.6.5. The set $\mathbb{Q}$ can be identified in a natural way with a subset of $\mathbb{Z} \times \mathbb{Z}$, using the map $m/n \mapsto (m, n)$ (when m/n is in lowest terms). It follows that $\mathbb{Q}$ is countable. □

EXAMPLE 1.6.6. The set of all lines in the plane that contain (at least) two points having integer coefficients is countable. For each line may be identified with the 4-tuple of integers coming from the coordinates of the two given points, and the set of such 4-tuples is countable. □

EXAMPLE 1.6.7. The set $\mathbb{Z} \times \mathbb{R}$ is uncountable, for it contains a copy of $\mathbb{R}$ by way of the map

$$\mathbb{R} \ni x \mapsto (0, x) \in \mathbb{Z} \times \mathbb{R} .$$

□

EXAMPLE 1.6.8. The set $\mathbb{C}$, the complex numbers, is uncountable. It contains a copy of $\mathbb{R}$ by way of the map

$$\mathbb{R} \ni x \mapsto x + i0 \in \mathbb{C} .$$

□

EXAMPLE 1.6.9. Let us now construct a concrete example of an uncountable set. Our example will be the set S of all sequences on the set $\{0, 1\}$, i.e., the set of all infinite sequences of 0s and 1s. To see that S is uncountable, assume the contrary. Then there is a first sequence

$$\mathcal{S}^1 = \{s_j^{(1)}\}_{j=1}^{\infty} ,$$

a second sequence

$$\mathcal{S}^2 = \{s_j^{(2)}\}_{j=1}^{\infty} ,$$

and so forth. This will be a complete enumeration of all the members of S. But now consider the sequence $\mathcal{T} = \{t_j\}_{j=1}^{\infty}$, which we construct as follows:

- If $s_1^{(1)} = 0$ then set $t_1 = 1$; if $s_1^{(1)} = 1$ then set $t_1 = 0$;
- If $s_2^{(2)} = 0$ then set $t_2 = 1$; if $s_2^{(2)} = 1$ then set $t_2 = 0$;
- If $s_3^{(3)} = 0$ then set $t_3 = 1$; if $s_3^{(3)} = 1$ then set $t_3 = 0$;

$$\cdots$$

- If $s_j^{(j)} = 0$ then set $t_j = 1$; if $s_j^{(j)} = 1$ then set $t_j = 0$;

etc.

Now the sequence $\mathcal{T}$ differs from the first sequence $\mathcal{S}^1$ in the first element: $t_1 \neq s_1^{(1)}$.

The sequence $\mathcal{T}$ differs from the second sequence $\mathcal{S}^2$ in the second element: $t_2 \neq s_2^{(2)}$.

And so on: the sequence $\mathcal{T}$ differs from the j^{th} sequence $\mathcal{S}^j$ in the j^{th} element: $t_j \neq s_j^{(j)}$. So the sequence $\mathcal{T}$ is not in the set S. But $\mathcal{T}$ is *supposed* to be in the set S because it is a sequence of 0s and 1s and all of these have been hypothesized to be enumerated.

This contradicts our assumption, so S must be uncountable. □

EXAMPLE 1.6.10. Consider the set of all decimal representations of numbers—both terminating and non-terminating. Here a terminating decimal is one of the form

$$27.43926$$

while a non-terminating decimal is one of the form

$$3.14159265\ldots.$$

In a non-terminating decimal, no repetition is implied; the decimal simply continues without cease.

The set of all those decimals containing only the digits 0 and 1 can be identified in a natural way with the set of sequences containing only 0 and 1 (just put commas between the digits). We just saw that the set of such sequences is uncountable.

Since the set of all decimal numbers is an even bigger set, it must be uncountable also.

As you may know, the set of all decimals identifies with the set of all real numbers. We find then that the set $\mathbb{R}$ of all real numbers is uncountable. (Contrast this with the situation for the rationals.) □

It is an important result of set theory (due to Cantor) that, given any set S, the set of all subsets of S (called the *power set* of S) has strictly greater cardinality than the set S itself. As a simple example, let $S = \{a, b, c\}$. Then the set of all subsets of S is

$$\Big\{ \emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\} \Big\} .$$

The set of all subsets has eight elements while the original set has just three. In general, if S has k elements then the power set of S will have 2^k elements.

Even more significant is the fact that if S is an infinite set then the set of all its subsets has greater cardinality than S itself. This is a famous theorem of Cantor, implying that there are infinite sets of arbitrarily large cardinality.

We conclude this discussion with a result that makes it easy to determine the cardinalities of many sets.

Theorem 1.6.11 ([Schroeder-Bernstein])**.** *Let A, B be sets. If there is a one-to-one function $f : A \to B$ and a one-to-one function $g : B \to A$, then A and B have the same cardinality.*