Fractions and Units in Everyday Life

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Fractions, in the form of percentages and rates, are pervasive in the workplace and in decision-making in one's personal life. However, it is in the transition from whole number arithmetic to fractions that too many students fall off the ladder of mathematical learning. They continue their education and become adults without ever understanding fractions.

Consider the following question on the TIMSS 8th grade test:

> Find the approximate value, to the closest integer, of the sum: 19/20 + 23/25.

Possible answers were a) 1, b) 2, c) 42, d) 45. (Answer: b) The majority of U.S. students chose c) or d). These students did not think of a fraction as a number. When asked to add two fractions and get an integer answer, they added the numerators or the denominators of the two fractions. The only numbers that they knew about were counting numbers (whole numbers). A fraction to them was some combination of two whole numbers. To be fair, fractions are a sophisticated mathematical concept compared to whole numbers.

The critical concept underlying fractions is units. By a unit, we mean a standard reference for measurement or counting. Units range from simple standards like inches and cents to more subtle standards such as the amount

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of sugar—say, 2/3rds of a cup of sugar—required in a recipe for a batch of brownies. Units also turn out to be the key to understanding multi-step, whole-number word problems. Being able to solve such word problems, along with using fractions, are the mathematical knowledge that a Business Roundtable task force (Schaar, 2005) has identified as essential for assembly line workers now employed in technologically oriented companies. Moreover, understanding fractions in the framework presented here is a critical foundation for quantitative reasoning in the modern world.

Whole number arithmetic was once all the mathematics that most people used in their jobs. Today, whole number arithmetic is performed in the workplace by machines for the sake of record keeping as well as accuracy. Employees no longer do arithmetic calculations themselves. Whole number arithmetic is still needed for simple mental calculations throughout daily life, but increasingly *its primary importance is as the mathematical foundation for future mathematical learning*.

Fractions have come to have a major role in the workplace. Whether on production lines or managers' desks, many of the numbers one encounters in business today are percents and rates—error rate, interest rate, employment rate, productivity level, etc. Thus all citizens today need to know how to use and interpret fractions. International comparisons like TIMSS reveal that too many U.S. students, in comparison to students in other countries, have trouble making the transition from whole number arithmetic to fractions. Efforts are underway to reorganize mathematics instruction in early grades to give greater attention to preparing students for fractions.

While instruction about fractions for college students, as part of a quantitative literacy curriculum, will obviously be different from K–8 instruction about fractions, a natural starting point for the former instruction is the extensive research about learning fractions in grades K–8 that has been undertaken by mathematics educators as well as several mathematicians. This essay attempts to summarize some of this research, (Post, 2002; Lamon, 2006; Steffe, Cobb, & von Glasersfeld, 1988; Wu, 2002), and suggests how that research can assist efforts to develop better understanding of fractions in college students. We also draw on the development of fractions in Singapore and Japan elementary mathematics curricula. This essay grew out of discussions at a workshop on teaching fractions at the Park City Mathematics Institute in July 2006.

Moving from whole numbers to fractions

Children develop an intuitive understanding of whole numbers in the context of counting objects. Because fractions and the arithmetic of fractions are much more complicated, intuition cannot be counted on to develop an understanding in a person's mind of what fractions are, much less how to calculate with them. Notation and terminology are much more important with fractions, but they can cause more problems than they solve.

For learners of all ages, definitions of basic mathematical concepts have to be framed with care: not too formal and not too informal. The common sense notion of a whole number as a counting number, used to count how many items are in a collection, provides a reasonable definition. In anticipation of rational and real numbers, whole numbers may later be identified with appropriate points on the number line.

On the other hand, most naïve approaches to understanding of a fraction, such as 1/3, can lead to misperceptions. Thinking of 1/3 in terms of a circle split into three thirds is a helpful place to start but can cause problems. A person with this image of 1/3 might forget that the pieces need to be equal and think of 1/3 as the name of one of the pieces when a circle split into 3 unequal pieces. This circle-based definition of 1/3 is not much help when one needs to find 1/3 of 24 pencils.

As soon as is judged feasible, a learner should be given the following definition of a fraction. This is the definition of a fraction used in many other c*ountries:*

> *A fraction is a number that is an integer multiple of some unit fraction.*

In mathematical notation, we mean a number of the form $k(1/l)$, for whole numbers k, l ($l > 0$). This definition assumes that the person has first developed a good understanding of what a unit fraction is. Unit fractions are discussed extensively in the next section. Note that in the essay, we will not worry about more complicated fractions, with numerators and denominators that are themselves fractions or irrational numbers.

Independently of formal study of fractions, people encounter fractions in a variety of everyday situations—telling time, making change, cooking recipes, sharing (when portions are not whole amounts), and measuring small lengths. There are many diverse day-to-day *interpretations* of fractions. For a fraction like 2/3, the two most common interpretations are

1. Representing two of three equally divided parts.
2. Representing the quantity resulting from a measurement, such as 2/3 meter.

Note that the first interpretation is based on an unspecified whole. A danger with multiple interpretations of fractions is that may be viewed as equivalent definitions of fractions, heightening confusion about what a fraction is.

Defining fractions in terms of unit fractions avoids a major conceptual problem, namely, establishing that a fraction is a number. That burden now falls to unit fractions. A second advantage of defining fractions in terms of unit fractions is that this approach separates the study of the numerator and the denominator of a fraction. Numerators are standard counting numbers, while denominators are a totally new quantity—they are units defined in terms of reciprocals. Again, this is the reason we focus in the rest of the article on unit fractions and units generally.

With so much time in early grades devoted to whole number arithmetic, students unconsciously reinforce their initial intuition that the term ‘number’ means only a ‘whole number’ or ‘counting number.’ A U.S. fourth grade student who has encountered fractions in measurement (time, money, lengths, etc.) and other contexts will still likely say that a fraction is not a number, but rather is a part of something. Many adults would probably say the same thing. The Rational Number Project devoted considerable effort to understanding the hurdles to learning fractions created by students’ belief that ‘number’ = ‘whole number.’ A similar problem arises with multiplication, which is initially learned as repeated addition, i.e., multiplication by a whole number. In this context, multiplication by a fraction makes no sense.

Because so many U.S. students never move beyond thinking of a number as a counting number, we should not be surprised that students mindlessly memorize operations with fractions in terms of the integers in the numerators and denominators of fractions without knowing what a fraction is or that they will assert that $1/3 + 1/5 = 1/8$.

On the other hand, a child’s first understanding of a number will necessarily be as a counting number and multiplication is naturally introduced as repeated addition. Thus the pedagogical goal must be *to help students extend, rather than abandon, these initial understandings of a number and multiplication*; Les Steffe calls this critical process *reconceptualization*. Students face this challenge over and over as they advance in their mathematical education.

Students do develop a valid understanding of fractions as numbers in practical settings. For example, people know that one fourth of a particular item (e.g., of a pie or a quart) plus two fourths of that item equals three fourths of the item. Further, these students can often find natural common units for adding fractions in familiar contexts, e.g., half an hour plus a third of an hour equals 50 minutes.

Teaching students of any age a true understanding of fractions can build on their experience with fractions as parts of something and their readiness to do simple calculation with fractions. However, it is a big step, with which education researchers continue to grapple, to go from thinking about fractions as parts of given objects, such as pies, to thinking about fractions as potential parts of an unspecified object.

We close this section by mentioning the ambiguity in the notation for fractions. The expression a/b, where a, b are whole numbers and $b > 0$, has two mathematical meanings. It is a rational number equal to the fraction $a(1/b)$. It is also a common way of writing the calculation $a \div b$. Students need to have a good understanding of fractions (the first interpretation) before the relationship between fractions and division (the second interpretation) is presented. If fractions are presented in the context of division, students can easily think of a fraction not as number in its own right, but rather as the quotient of whole number division. In this flawed framework, it makes sense to learn arithmetic operations on fractions by memorizing integer-valued formulas for the resulting numerators and denominators. On the other hand, division naturally arises early in the discussion of fractions, e.g., finding 1/4 of some collection, such as 12 eggs. How division is best connected with learning fractions is an open challenge.

Unit fractions

Unit fractions, such as ¼, are a natural precursor to fractions. Unit fractions arise frequently in day-to-day conversations—a quarter (the coin), a quarter after 5 o'clock, a quarter of a mile down the road, a quarter of a cup of flour, a 1 ¼ inch screw, etc. A growing number of U.S. mathematics textbooks now discuss unit fractions to varying degrees starting in first grade.

For young children, unit fractions evolve from counting numbers: a pie divided into fourths is split into equal pieces which when counted amount to 4. Given two pies divided into sixths with 3 sixths left in the first pie (the three other sixths were eaten) and 2 sixths left in the second pie, first-grade students can count, and later add, the sixths in the two pies to obtain a total of 5 sixths.

Here is an activity (suggested by Yale mathematician Roger Howe) emphasizing the difference between counting numbers and unit fractions. Given a pitcher with a capacity of one quart of water, a student can determine the capacity, say 4 quarts, of a second pitcher by counting how many quart-size pitcherfuls it takes to fill up the new pitcher. Now consider a third pitcher whose capacity is 1/4th of a quart. To determine what this unit-fraction is, the student needs to determine how many pitcherfuls of the third pitcher it takes to fill the quart pitcher.

As noted above, while it natural to use pictures of pies or some other common geometric figures when starting to work with unit fractions, there are several misconceptions that can arise from such geometric examples of unit fractions. Extensive examples with measurement and with dividing a collection of items in equal shares can help students develop a more general mental understanding of unit fractions.

There are a number of different steps that extend this basic start with fractions. Intermixed with pictorial problems about unit fractions can be an occasional set of purely numerical problems (no figures) involving simple addition of fractions, such as 2/5 + 2/5. Another step is to have problems whose answers are improper fractions and then to restate an answer such as 5/4 cups of sugar as 1¼ cups of sugar. A parallel step is to have answers that are whole numbers, such as 4 fourths or 8 fourths, and to convert from fourths to whole numbers. Likewise, one can do conversions from whole numbers to unit fractions. Simple examples of multiplication and division of amounts stated in unit fractions can be introduced, e.g., given a recipe requiring 2 fourths of a cup of sugar, how many batches of the recipe can we make with two cups of sugar. Initially these problems would be accompanied with diagrams to help organize students' thinking.

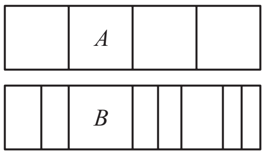
These arithmetic experiences are reinforced and extended by the use of unit fractions in measurement problems. It is important that all whole number arithmetic be interpreted in terms of measuring lengths, e.g., addition is concatenation of lengths; multiplication is repeated concatenation of lengths. Thus the number line has a natural interpretation as distances from 0 (the start point). Unit fractions arise naturally in measuring lengths. Unit fractions also have a natural role in the measurement of time, money, and later area and volume. Note that the transition from multiplication by whole numbers, i.e., repeated addition, to multiplication by fractions is a natural extension in linear measurements: if bricks are 8 inches long, how would long would a row of 3½ bricks be?

While standard rulers are subdivided into halves, fourths, eighths, and sometimes sixteenths of an inch, students should have access to rulers with different types of subdivisions, e.g., in 5ths and in 10ths of an inch. In measuring lengths, students are initially equating whole numbers with lengths. Over time, it becomes natural to view all lengths as numbers, and an intuitive feeling for numbers as points on the number line develops. This is why Berkeley mathematician Hung-Hsi Wu likes to define fractions in terms of the number line.

When fractions are discussed in a collegiate quantitative literacy course, students are unlikely to be familiar with the definition of a fraction as a multiple of a unit fraction. The newness of this approach can be an excuse to review quickly several steps in the development sketched in the preceding paragraphs. It is important constantly to pose unit fractions problems in applied settings so that the 'unit' in 'unit fraction' has meaning. An example is how high will a pile of 4 notebooks be if each notebook is 5/8″ thick. The answer would first be found in terms of 1/8ths and then converted to whole inches. A more advanced, inverse version of this problem would be, how many notebooks that are 5/8″ thick can be piled into a box that is 2½ feet deep.

We noted above that unit fractions are derived in students' minds from counting numbers as follows: a pie divided into fourths is split into equal pieces which count to 4. To give a sense of the cognitive challenge students face in moving beyond this image of unit fractions, we cite a scene from a demonstration class of fifth graders led by Deborah Ball at the Park City Mathematics Institute in summer 2006. When students were asked to go to the blackboard and highlight 1/8th of a collection of 24 circles that had been drawn, one student first divided 8 into 24 to get 3, and then he proceeded to partition the set of 24 circles into groups of 3. He had to check that 8 groups of 3 balls completely partitioned the set of 24 balls before being able to say that 3 balls were 1/8 of the set of 24 balls. That is, the concept of 1/8 of a something, and implicitly the general concept of a unit fraction, did not exist in his thinking. He only could conceive of dividing something into 8 equal parts, a concept based on counting numbers.

Here is another example of the trouble that students have in moving beyond the equal division model (Tzur, 2006). Consider the two rectangles on the right, both with the same dimensions. The upper one is divided into 4 equal sections. The lower one is divided into 8 unequal sections. We are told that section A in the upper rectangle is the same size as section B in the lower rectangle. The question is, what fraction of the lower rectangle is section B?



Many middle school students and their teachers will assert that section B is our fourth of the upper rectangle but that one cannot tell what fraction it is of the lower rectangle. Many adults may have the same problem. This example shows the limitations of pictorial models of fractions.

Units

To illustrate the role of units in working with fractions, consider the following problem:

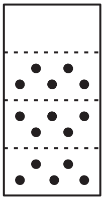
> Some balls are taken from a box and 15 balls are left. This number 15 is three quarters of the number of balls that started in the box. How many balls started in the box?

The reasoning for solving this problem involves two types of units. The problem can be restated: if we know 3 fourths of a quantity, what is 4 fourths of the quantity. The key is to think in terms of fourths. If one fourth is our unit, then the problem comes, if three units equal 15, what do four units equal. The natural intermediate step in the solution is to determine what one unit equals. We get that one unit is $15 \div 3 = 5$, balls, and the boxful of 4 units equals $4 \times 5 = 20$ balls.

While fourths were the units for initially analyzing the problem, 5's were the units involved in determining the final answer. One could say that one unit equals our fourth of a boxful, and then restate that unit as equal to 5 balls. One could also look at these two units as a ratio: 5 balls per fourth of a boxful. Analyzing relationships between two or more units underlies the solution of almost all real-world problems involving fractions. Many educators refer to the (implicit or explicit) use of units to solve such a problem as multiplication reasoning. Such reasoning is a prerequisite to solving fraction problems.

The problem could also be modeled algebraically as $(3/4)x = 15$ and solved for x to obtain $x = 15 \div (3/4)$, with the right-hand side computed with the invert-and-multiply rule for division by fractions. That rule, of course, yields the same calculation as in the previous analysis: divide 15 by 3 and multiply the result by 4 (or the order could be inverted). It is preferable that students be able to perform the reasoning described above than that they memorize (and soon forget) the invert-and-multiply rule for fraction division.

One learning aid heavily used in East Asian countries is diagrams. The 1999 TIMSS Video Study (NCES, 2003) found that 83% of the problems in 8th grade mathematics lessons in Japan used diagrams or drawings while the percentage in the U.S. was just 26%. For example, when problems like the one above are first encountered, students would see a diagram like the one on the right to point them towards the solution. College students who are being reacquainted with fractions should be asked to draw similar diagrams to help their initial reasoning.



Many rate problems have a similar structure to the problem about balls in a box. For example:

> If a car going at a constant speed covers 48 miles in ¾ of an hour, how far will it go in one hour? Or equivalently, how fast is it going (in miles per hour)?

To solve this we must first focus on measuring time in fourths of an hour. Then we switch to the dual unit of 16 miles, the distance traveled in a fourth of an hour.

A nice grammatical analogy, often attributed to Ken and Herb Gross, is sometimes helpful for understanding the relationship between numbers and units. They call numbers 'adjectives' and they initially use these 'adjectives' only in the context of modifying a 'noun' such as 5 pencils or 2/3rds of a pie. The nouns are then extended to include units of measurement and units defined in terms of other adjective-noun pairs such as 4 (boxes of 500 pencils) and 5 (eighths of an inch).

Converting between units

One of the critical mathematical building blocks for working with fractions is equivalent fractions, different fractions that represent the same rational number, e.g., ½ or 2/4 or 5/10 or 13/26, etc. However, the general topic of equivalent representations of a quantity arises repeatedly in measurement problems, e.g., ½ foot = 6 inches, or 50 cents = 10 nickels = 5 dimes = 2 quarters = ½ dollar, as does the issue of finding a new, common representation for adding quantities in different units, e.g., adding 1/3 foot + ¼ foot by converting to inches, or adding 2 dimes and 1 quarter by converting to cents.

Equivalent fractions are a particular case of a more general mathematics topic, namely converting a number expressed in terms of one unit to another unit. Finding a new unit for representing different quantities arises in word problems involving multiplication and division.

Consider the problem:

> A brick is 8 inches long. How many bricks must be placed end to end to reach 10 feet?

First we express the length of 10 feet in terms of inches—120 inches—using the conversion rule 1 foot = 12 inches. This is the first change of units. Then we convert the length in inches into another unit, brick lengths, using the conversion rule 1 brick length = 8 inches. The first conversion involved a multiplication and the second a division.

The following solution strategy follows the spirit of unit fraction examples in the previous section. After noting that one brick length is 2/3rds of a foot, one converts the total length from feet to 1/3rds of a foot. This is an easy conversion—multiply by 3—to keep straight in one's mind, and work with unit fractions continually reinforces such conversion strategies. So now the length is 30 1/3rds of a foot. Since each brick is 2 1/3rds of a foot long, we need 30/2 = 15 bricks.

The roles of the units can be inverted. We look at the problem in terms of brick lengths per foot: 1 foot = 1½ brick lengths. Then 10 feet = 1½ × 10 = 15 brick lengths. This problem illustrates the fact that *any time we perform multiplication or division in solving an applied problem, we are explicitly or* *implicitly converting units*. Moreover, there are a number of choices for units. In this case, inches, feet, and brick-lengths.

Let us recast this problem in a business setting. Given that an adult pays $40 for admission to an amusement park, what level of attendance is needed to generate $100,000 in a day? Answer: 100,000/40 = 2,500 adults. If one wanted to plan for the level of services needed in the park and the cost of these services, then one would probably find it useful to think simultaneously in terms of multiples of 2,500 people (demand) and $100,000 (available income). A natural extension of this problem would incorporate the fact that children pay, say, $30 for admission. Now many ratios of income and expenses come into play in analyzing demands and income from various mixes of adults and children.

Let us next consider a word problem involving three units. (It is the first word problem to appear in the 5th grade Singapore mathematics textbook (Singapore Math, 1997)):

> Mrs. Li bought 420 mangoes for $378. She packed them into packets of 4 mangoes each and sold all the mangoes at $6 per packet. How much money did she make?

The initial units that appear in the problem statement are mangoes and dollars. Later in the problem statement, packets enter. We need to convert units for measuring mangoes from individual mangoes to packets of 4 mangoes. Given that 4 mangoes go into packet, we divide 420 by 4 to obtain 105 packets. Now we convert our units for measuring mangoes from packets to value in dollars. The conversion factor is that one packet yields $6 dollars, and so we multiply for this conversion to obtain a value of 105 × $6 = $630. Finally, we have cost and income in the comparable units, dollars, and so the amount of money made in this activity, $630 – $378, can be computed.

Another way to approach this problem is to look for a way to convert directly from units of mangoes to units of money. This conversion requires determining a rate of income per mango. Since 4 mangoes in a packet sell for $6, we obtain a rate of $6/4 (= $1½ per) per mango.

This problem illustrates the fact that c*alculation with a fraction can frequently be recast as a short cut for a two-step calculation involving a multiplication and a division with whole numbers*.

Let us now use units-based reasoning to analyze the following problem of fraction multiplication: 2/3 × 4/5 = ?. Interpreting 2/3 as 2 thirds [=2(1/3)], we first need to find 1/3 of 4 fifths. We are initially stuck because 1/3 of 4 is not a whole number. We change to a new unit that is sure to work, namely 1/(3 × 5). So we convert 4 fifths to 12 fifteenths [= 12(1/15)]. We can find 1/3 of 12 fifteenths by dividing 12 by 3; it is 4 fifteenths (4/15). Finally we multiply this amount by 2 to find $2 \times [4(1/15] = 8/15$. Diagrams can help with this problem. For example, 4/5 could initially be depicted with a rectangle partitioned by horizontal lines into 5 equals parts with the lower four parts shadowed. Then the rectangle could be subdivided with 3 vertical lines into 15 equals parts. One third of the 12 shadowed parts is found, etc.

Students' knowledge about units can also be used to revisit whole number addition and subtraction and to appreciate the role of conversion among decimal units in the standard algorithms of arithmetic. The place value notation is now seen as a system of related decimal units. The key steps of carrying in addition and borrowing in subtraction involve converting between consecutive decimal units. The standard multiplication and division algorithms can be studied in terms of how they combine partial computations in different decimal units.

We conclude this section with an important subtlety about the role of units in division. People often say that division is the 'inverse' operation of multiplication. However, there are two very distinct interpretations of how division is the 'inverse' of multiplication. Interpreting multiplication as repeated addition, the equation $4 \times 5 = 20$ says that the sum of four 5's is 20. Inverting this process, $20 \div 5 = 4$ could be interpreted as saying that 4 is the number of 5's that need to be summed to get 20. What then is the interpretation of $20 \div 4 = 5$ in terms of the multiplication $4 \times 5 = 20$? It is, what number when summed 4 times yields 20. A more familiar way to state this is, when we divide 20 into 4 equal parts, what is the size of each part. The first problem $20 \div 5 = 4$ was a change of units: we count numbers by 5's instead of by 1rs. The second problem is a partitioning situation, although it can also be interpreted with a change of units as follows: what should the units be if we want to 4 units to equal 20.

Concluding remarks

In this essay I have tried to make the case that understanding fractions well by relating them to units is both important and intellectually rich. It is definitely worthy of a college level course in quantitative literacy. More generally, fractions are a much richer mathematical construct than most people realize. This complexity is reflected in the fact that mastery of fractions was not normally required for university admission just 100 years ago, only whole number arithmetic was required (DeTurck, 2000).

However, today fractions arise frequently in daily life as percentages, rates and proportions. The details of teaching these applications of fractions to college students have been well explored by many others in the quantitative literacy movement. What is not as well appreciated is their connection to multi-step, whole-number word problems, as presented in this essay.

We note that greater attention to units also brings mathematics instruction closer to science instruction, where units play such an important role.

We close with a concrete example of the challenges in implementing the program outlined above. We refer again to the Park City model class of Deborah Ball's where students were asked to find one eighth of 24 balls drawn on the blackboard. One student divided 8 into 24, and, based on his answer of 3, partitioned the 24 balls into 3 groups of 8 each. Next he marked one ball in the first group of 8. However, the student then stopped and gave 1 as the answer. A cognitive specialist watching the students speculated what had gone wrong. Like many other students of his age, this student had trouble keeping track of more than two units at one time. He reorganized the problem of finding 1/8th of the whole group of 24 by first breaking 24 into 3 groups (units) of 8's. He then determined what 1/8th of a group of 8 was, but had lost track of the relationship between the original group of 24 and the group of 8.

Keeping track of multiple units is an example of a critical cognitive skill that mathematicians generally take for granted. Thus, to better prepare students to learn fractions, one needs not only to understand the proper mathematical development of underlying concepts, such as units, but also to understand the hurdles that students face when they try to learn these concepts.

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