Representing homology classes of 4-manifolds

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The setting

Let $W = W^4$ be a 4-dimensional manifold, usually simply connected, always piecewise linear (PL) and orientable.

The key to understanding the topology of 4-dimensional manifolds is to understand the embeddings of 2-dimensional surfaces in W.

These particular dimensions are special:

- dimension 2 is the middle dimension (where we expect minimal selfintersections)
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The homology group

The second homology group $H_2(W;\mathbb{Z})$ carries the algebraic information about surfaces in W.
Consider $\alpha \in H_2(W;\mathbb{Z})$. We want to represent α in a geometrically useful way.

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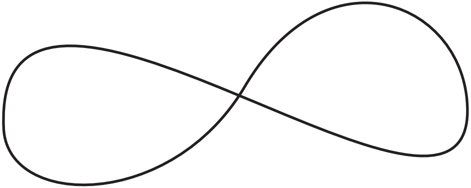

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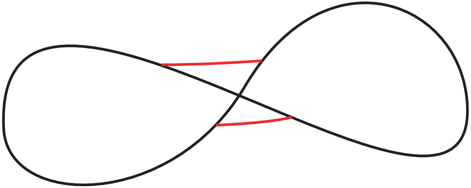

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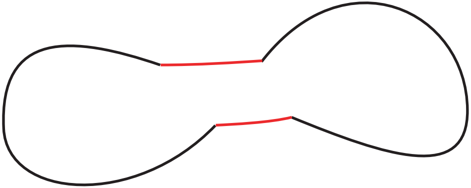

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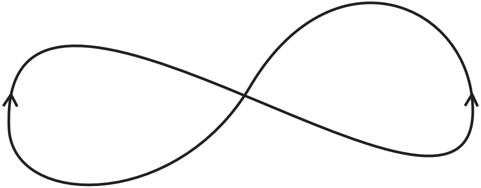

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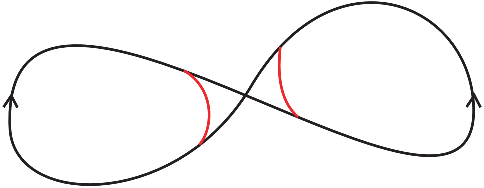

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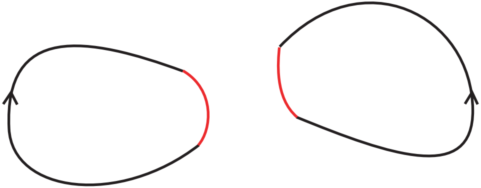
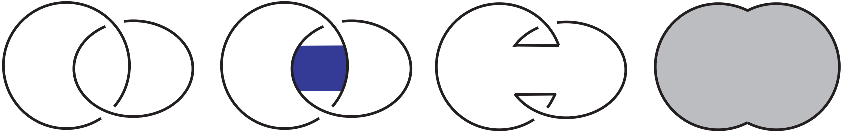
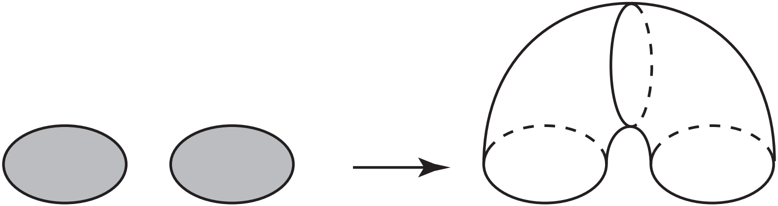


Constructing an embedded orientable surface

In a neighborhood of a crossing point, remove a pair of transverse disks and sew in an annulus.



The picture in the target.



The picture in the source.

The main question

This talk will explore the following question.

Main Question

Assume W is simply connected. Then can any 2-dimensional homology class α be represented by a PL embedded 2-sphere?

Another way to formulate the question: Is any continuous map $f : S^2 \to W$ homotopic to a PL embedding?

The compact and noncompact cases will be considered separately. First it is necessary to have a good way to describe a 4-manifold. One convenient way to describe a compact PL manifold is to decompose it into the union of a finite number of "handles."

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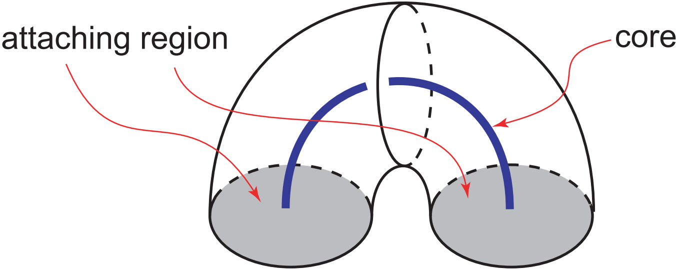
Definition of a handle

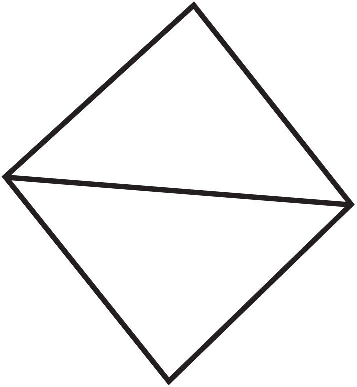
Let W be an n-dimensional manifold with nonempty boundary. An *n-dimensional handle of index k* attached to W is an n-cell H such that there is a homeomorphism $h : I^k \times I^{n-k} \to H$ with $H \cap W = h(\partial I^k \times I^{n-k})$. The *core* of H is $h(I^k \times \{0\})$.



A 3-dimensional 1-handle

Handle decompositions of manifolds

Every compact PL manifold naturally decomposes as a union of finitely many handles of increasing index.



Start with a triangulation of the manifold.

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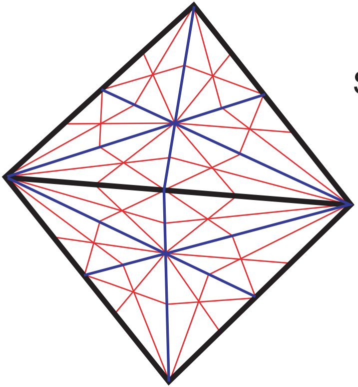

Subdivide twice.

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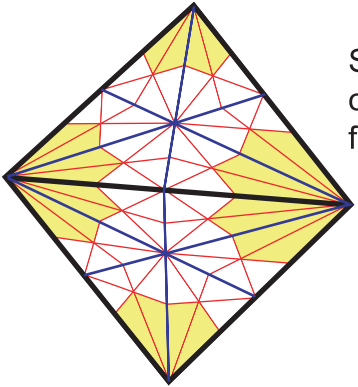

Simplicial neighborhoods of the original vertices form 0-handles.

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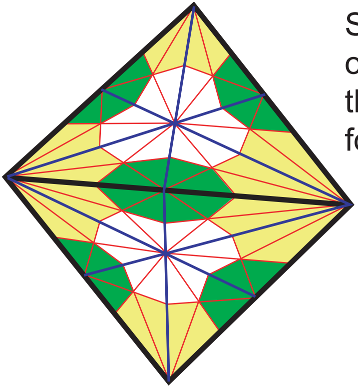

Simplicial neighborhoods of the barycenters of the original 1-simplices form 1-handles, etc.

The Kirby calculus

R. Kirby has developed a notation that allows a visual description of a connected 4-manifold with only handles of index ≤ 2.

- Consolidate all the 0-handles into one. The 0-handle is a 4-ball and subsequent handles are attached to its boundary, which is S^3.
- Adding a 1-handle is equivalent to removing a 2-handle, so each 1-handle is represented by an unknotted circle.
- A 2-handle is represented by an attaching curve with a designated framing. The 2-handle H is parametrized by $h : I^2 \times I^2 \to H$ and the framing is $\mathrm{lk}(h(\partial I^2 \times 0), h(\partial I^2 \times x))$ for some $x \neq 0$.

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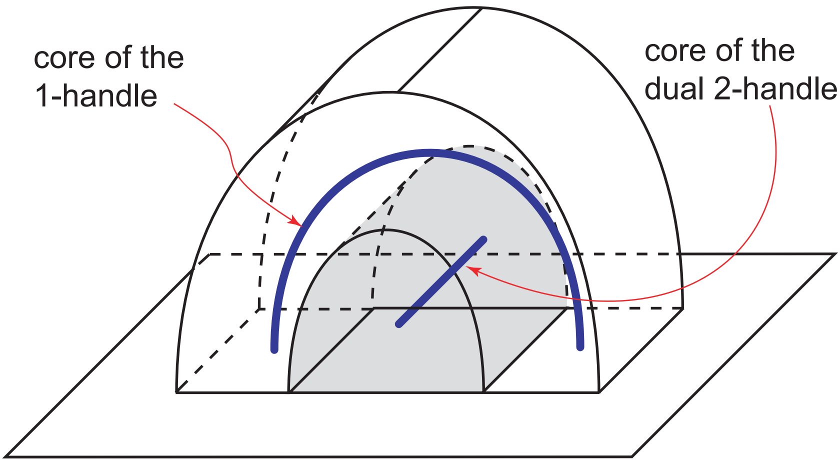
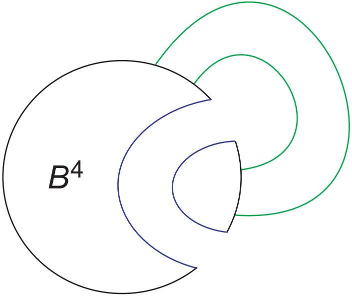
Adding a 1-handle = Removing a 2-handle

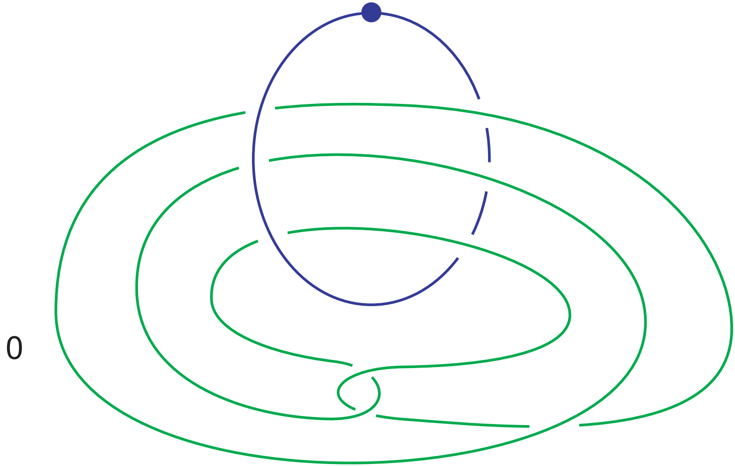


Schematic diagram



In this schematic diagram the dimensions are divided by two. A 2-handle has been subtracted and a 2-handle has been added. Note that in dimension $n = 2$, removing a 1-handle is equivalent to adding a 0-handle.

Example: the Mazur manifold



Mazur-like contractible 4-manifolds

Definition. A *Mazur-like contractible 4-manifold* is a contractible 4-manifold having a handle decomposition satisfying the following conditions.

- One 0-handle.
- A finite number of 1-handles.
- The attaching curve for the *i*-th 2-handle is homotopic to the curve that goes exactly once over the *i*-th 1-handle and and does not go over the other 1-handles.
- No handles of index ≥ 3.

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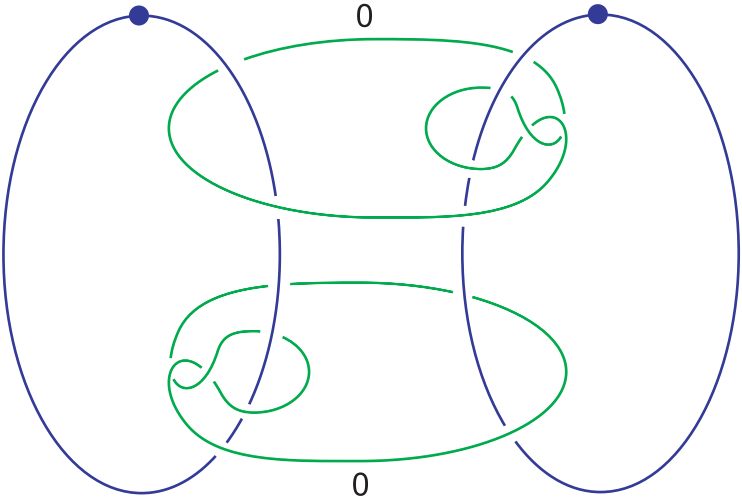


The structure theorem

We now return to the problem of representing a homology class. The following theorem allows α to be represented by a much simpler 4-manifold and reduces the main question to a special case.

The Structure Theorem for Compact Manifolds

If W is a compact, simply connected PL 4-manifold, then each element of $H_2(W;\mathbb{Z})$ can be represented by a compact 4-dimensional PL submanifold $M \subset W$ such that M consists of a Mazur-like contractible 4-manifold with a single 2-handle attached.

Related questions

The Structure Theorem does not directly solve the Main Problem, but it does reduce it to a simpler special case.

Matsumoto Question

Is every homotopy equivalence $S^2 \to W$ homotopic to an embedding?

Another consequence of the Structure Theorem is that the Main Problem would have a positive solution if the following question had a positive answer.

Zeeman Question

If V is a compact, contractible 4-manifold, does every loop on the boundary of V bound a disk in V?

Submanifolds of S^4

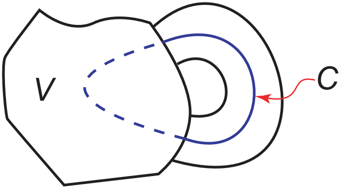
The Structure Theorem does allow a complete solution of the Main Problem in one important special case.

Corollary (of the proof)

If W is a compact, simply connected PL submanifold of S^4, then every element of $H_2(W; \mathbb{Z})$ can be represented by a locally flat topological embedding of S^2.

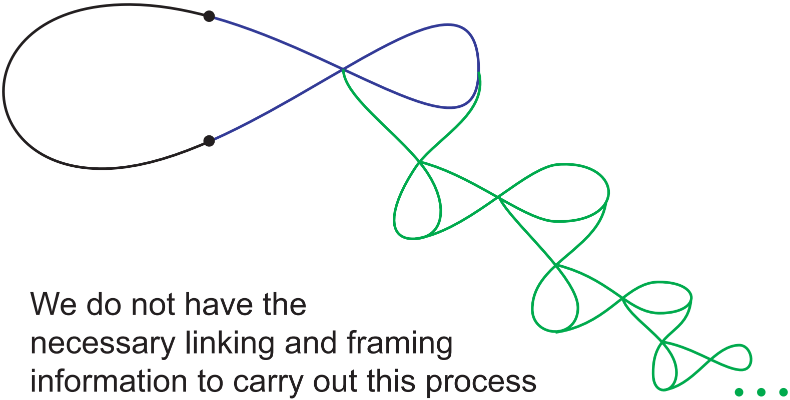
Proof of the Corollary

Use the Structure Theorem to represent the given element of $H_2(W;\mathbb{Z})$ by a PL submanifold of the form $M = V \cup h^{(2)}$. The proof of the theorem indicates that V can be constructed to be π_1-negligible. By Freedman there is a homeomorphism $\phi : V \to S^4 - V$ that is the identity on ∂V. The 2-sphere we seek is $C \cup \phi(C)$, where C is the core of the 2-handle.



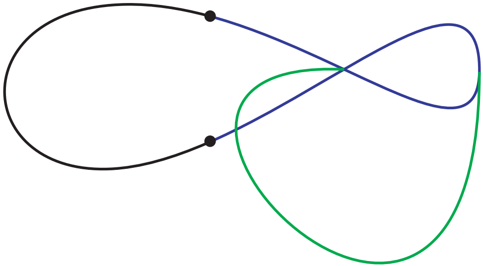
An infinite construction

The Casson-Freedman construction



We do not have the necessary linking and framing information to carry out this process

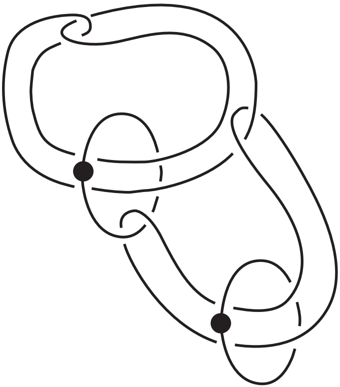
Another infinite construction



Desingularize the attached disk at the expense of introducing a point of intersection with the original disk.

A Kirby diagram of the construction

First stage handles



Second stage handles

Negative result in noncompact case

If the process on the previous slide is continued, the result is a noncompact manifold W having the following properties.

- W is an open subset of S^4.
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- There is no compact submanifold $C \subset W$ such that the inclusion $C \hookrightarrow W$ is a homology equivalence.

Hence the structure theorem fails for noncompact manifolds.

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Comparison with dimension three

The Scott Core Theorem fails in dimension four.

Theorem (P. Scott)

Given a 3-manifold W (not necessarily compact) with finitely generated fundamental group, there is a compact three-dimensional submanifold $C \subset W$ such that $C \hookrightarrow W$ induces an isomorphism on fundamental groups.

The 4-dimensional example does not contain a compact submanifold such that the inclusion induces an isomorphism on first and second homology.

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Comparison with codimension three

The noncompact example is not essentially 4-dimensional, but is a codimension-two phenomenon. The process of attaching contractible objects to the singular set to eliminate it works out because the dimensions of intersections get progressively smaller.

Theorem (Stallings)

Let $f : K^k \to M^n$ be a map of a compact polyhedron into a PL manifold. If

- $k \leq n - 3$*, and*
- f *is* $(2k - n + 1)$*-connected.*

then there exists a subpolyhedron $L \subset M$ and a simple homotopy equivalence $g : K \to L$ such that g and f are homotopic in M.

Proof of the Structure Theorem

Take a triangulation of W that includes $f(S^2)$ as a subcomplex. Let N be the union of the handles that intersect $f(S^2)$. Consider the region $\overline{W - N}$. It is filled with handles. Cancel all handles of index 0 and 4. Trade the 2-handles for 3-handles (standard technique). This leaves only handles of index two and three. The handles that look like 3-handles when viewed from the perspective of N look like 2-handles when viewed from the perspective of ∂W.