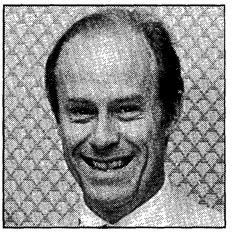
Things I Have Learned at the AP Reading

Dan Kennedy



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It almost goes without saying among those who have participated in the process that the benefits of the Advanced Placement reading are many and varied. There is, for example, the practical benefit of getting 161,000 exams graded in a smooth and professional manner. For AP readers, there are also the social benefits of seeing old friends and making new acquaintances with teaching colleagues from across the country. There are the nutritional benefits of storing up the valuable reserves of fat and cholesterol that their bodies need to make it through the long, hot summer. But one of the most important professional benefits for AP readers is the opportunity for calculus teachers to learn more about calculus, and that is the aspect of the reading that I would like to reflect upon in this paper.

Only the most naive of God's creatures—for example, our students—would assume that their teachers know everything about the subjects that they teach. We who teach calculus have learned to be particularly humble in that respect, encountering fairly regularly questions that at least cause us to tell our students, "Say, that's a good question, but let's not take class time on that right now; I'll explain it to you tomorrow." This buys us enough time, we hope, to solve the problem in private, where we can consult the solution manual.

We have all learned a lot of calculus that way. But those of us who have read AP calculus papers over the years have also learned that the simplest of problems can have remarkable subtleties buried just beneath the surface. Such subtleties might go undetected under normal circumstances, but under the abnormal circumstances of thousands of panicky students thrashing around with unfamiliar mathematical tools, they are soon exposed and demand our scrutiny. It is in these unexpected moments that I have learned some fascinating calculus facts over the years, and I would like to share a few of them with you in this paper. I hope that some of them will give you the same thrill of discovery that they gave me.

1. Radical Lies That We Tell Students in Algebra

This was AB-1 in 1988:

1. Let f be the function given by $f(x)=\sqrt{x^4-16x^2}$.
   a) Find the domain of f.
   b) Describe the symmetry, if any, of the graph of f.
   c) Find $f'(x)$.
   d) Find the slope of the line *normal* to the graph of f at $x=5$.

This is a pretty harmless-looking problem, but imagine your best algebra student looking at that radical and thinking, "Uh-oh. Better simplify that thing before I go on." We have almost programmed our students to think that way. So, the student writes

$$\sqrt{x^4 - 16x^2} = |x|\sqrt{x^2 - 16}\,.$$

(Notice the absolute value. I said that we were imagining your *best* algebra student.)

Now, how does the student answer part (a)? If she uses the simplified expression, she sets $x^2 - 16 \geq 0$ and arrives at a domain of $(-\infty, -4] \cup [4, \infty)$, losing the domain point {0} (and, of course, a point from her AB-1 score). If she uses the unsimplified expression, she sets $x^4 - 16x^2 \geq 0$ and arrives at the correct domain of $(-\infty, -4] \cup \{0\} \cup [4, \infty)$. (Actually, many AB students that year divided by x^2 and arrived at the wrong domain anyway, but your best algebra student would surely not have done *that*.)

I will leave it to the reader's imagination to consider what happened to the absolute value crowd when they moved on to part (c).

There are many examples of expressions that *pick up* domain values when simplified, such as $\frac{x^2-4}{x-2}$ or $\frac{\sin x}{\tan x}$, but this problem provides a rare example of an expression that *loses* a domain value when simplified. The dozens of AB students who lost that point in 1988 were probably not as charmed by this subtlety as we were, but it made for some nice conversation among teachers that year. We all resolved to be a little more careful to think about equations like

$$\sqrt{x^4 - 16x^2} = |x|\sqrt{x^2 - 16}$$

as *identities*. Identities have domains of validity, and this one is invalid at $x = 0$.

2. Proving Differentiability at a Point

There have been several "split-function" questions over the years, such as BC-4 in 1992:

4. Let f be a function defined by $f(x) = \begin{cases} 2x - x^2 & \text{for } x \leq 1, \\ x^2 + kx + p & \text{for } x > 1. \end{cases}$
   (a) For what values of k and p will f be continuous and differentiable at $x = 1$?
   (b) For the values of k and p found in part (a), on what intervals is f increasing?
   (c) Using the values of k and p found in part (a), find all points of inflection of the graph of f. Support your conclusion.

To solve part (a), the typical good student will paste together the two sides of the function at $x = 1$ to make it continuous

$$2(1) - 1^2 = 1^2 + k(1) + p \Rightarrow k + p = 0$$

and then paste together the derivatives of the two sides at $x = 1$ to make it differentiable

$$2 - 2(1) = 2(1) + k \Rightarrow k = -2.$$

The two conditions taken together imply that $k = -2$ and $p = 2$.

This is all well and good, but it leads many students (and once led me) to conclude that this is how to show that a function is differentiable at a point: establish continuity and show that the derivative is the same coming in from the left as it is coming in from the right. Luckily for everyone doing BC-4, that approach *will* successfully establish differentiability *if* it works. But it might *not* work, as was shown in 1982 on BC-7:

7. Let f be the function defined by $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$

   (a) Using the definition of the derivative, prove that f is differentiable at $x = 0$.

   (b) Find $f'(x)$ for $x \neq 0$.

   (c) Show that f' is not continuous at $x = 0$.

We solve the problem as follows:

(a) $f'(0) = \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{h^2 \sin(1/h) - 0}{h} = \lim\limits_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0.$

(b) $f'(x) = 2x \sin\left(\frac{1}{x}\right) + x^2 \cos\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) = 2x \sin\left(\frac{1}{x}\right) - -\cos\left(\frac{1}{x}\right).$

(c) Because the function $\cos\left(\frac{1}{x}\right)$ diverges by oscillation on both sides of 0, it follows that $\lim\limits_{x \to 0} f'(x) = \lim\limits_{x \to 0} \left(2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right)$ does not exist. Therefore, f' is not continuous at $x = 0$.

The implication of this problem is subtle, but profound. Notice that $f'(x)$ *does* exist at $x = 0$, but that this fact could not be discovered by looking at $f'(x)$ to the left and right of $x = 0$.

This same problem also taught me the true meaning of right- and left-hand derivatives. Notice that f is differentiable at 0, so both its right- and left-hand derivatives exist there. They are the two limits $\lim\limits_{h \to 0^-} \dfrac{h^2 \sin(1/h) - 0}{h}$ and $\lim\limits_{h \to 0^+} \dfrac{h^2 \sin(1/h) - 0}{h}$, both of which equal 0. They are *not* the same as $\lim\limits_{x \to 0^-} f'(x)$ and $\lim\limits_{x \to 0^+} f'(x)$, both of which fail to exist.

Students could have made perfect scores on these problems and never realized half of what I just said, but because I graded their papers at the reading those same problems afforded me the chance to learn those nuggets of calculus from my colleagues.

I later learned other interesting things about this particular split function, which is shown in Figure 1. As we have just seen, this function is both continuous and differentiable at $x = 0$. The graph of

$$g(x) = \begin{cases} x \sin\left(\dfrac{1}{x}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$$

is also shown in Figure 1. This function is continuous but it is an easy exercise to show that it is *not* differentiable at $x = 0$. You can almost see why, too: you could "zoom in" endlessly on the right-hand graph at the origin and the graph would get no straighter than it is now, whereas if you were to zoom in on the left-hand graph

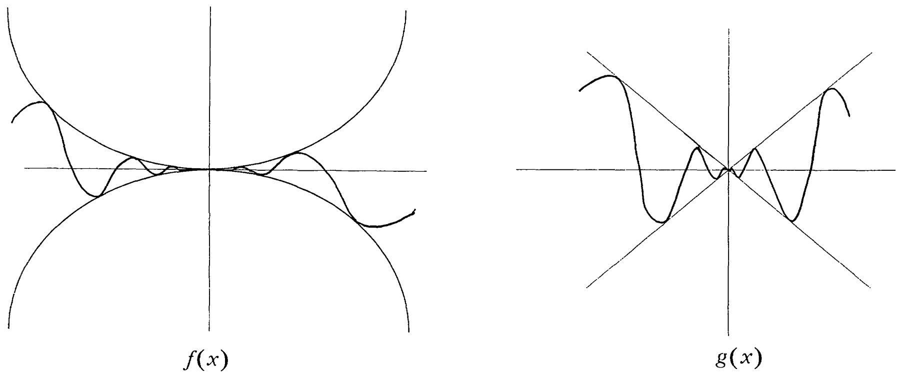
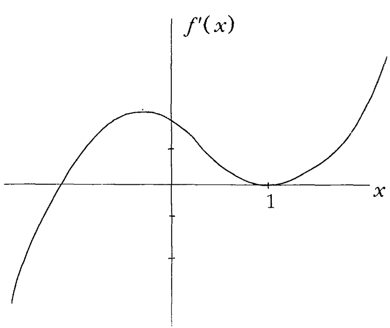


Figure 1

the curve would get flatter and flatter at the origin. If you find that your eye cannot quite distinguish why both $\lim_{x\to 0^-} f'(x)$ and $\lim_{x\to 0^+} f'(x)$ fail to exist in the left-hand graph, welcome to the club. This is why there will always be the need for algebraic limits in first-year calculus!

3. The First Derivative Test for Points of Inflection

There have been so many questions on AP exams over the years asking students to justify points of inflection that it is impossible to say when this issue first arose, so I will put it in the context of the following hypothetical question:



The expected response is that there is a point of inflection because there is a turning point of the graph of $y = f'(x)$ at $x = 1$ (or some equivalent statement about f'' changing sign), but what would you do with a student who says the following:

There is a point of inflection at $x = 1$ because $f'(1) = 0$ while $f'(x)$ is positive on either side of $x = 1$.

This student has noted the "shelf point" at $x = 1$ and is arguing that it must be a point of inflection, as in the familiar graph of $y = x^3$. There is no mention of the sign of the second derivative; indeed, the whole argument hinges on the sign of the first derivative. Does this student get credit for a valid justification?

Since this response actually occurred one year, the table leaders had to decide whether this was a valid theorem. The smart money being on "no" from the outset, most of their efforts were concentrated on finding a counterexample. My notes do not record who came up with one first, but later conversations have variously attributed it to Robert Ellis, Tom Tucker, or Bruce Peterson. (Bruce, chair of the Committee in those days, actually published a paper on the topic.) This is the function that I found in my notes from the reading:

$$f(x) = \begin{cases} 12x^5\sin(1/x) + x^3 & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

We can show that $f'(0) = 0$ and that $f'(x)$ is positive on both sides of $x = 0$, and yet the graph of f does not have a point of inflection at the origin. The salient feature of the function is that its graph changes concavity infinitely often in every neighborhood of zero. Here is a sketch of the proof:

1. (Proof that $f'(0) = 0$):

$$f'(0) = \lim_{h \to 0} \frac{12h^5\sin(1/h) + h^3 - 0}{h}$$

$$= \lim_{h \to 0} 12h^4\sin(1/h) + h^2 = 0 \cdot (\text{bounded}) + 0 = 0.$$

2. (Proof that $f'(x) > 0$ in some neighborhood of 0):

$$f'(x) = 60x^4\sin(1/x) + 12x^5\cos(1/x) \cdot (-1/x^2) + 3x^2$$

$$= x^2\left(60x^2\sin(1/x) - 12x\cos(1/x) + 3\right)$$

$$\downarrow \quad \Downarrow \quad \downarrow \quad \Downarrow \quad \downarrow$$

$$0 \quad (\text{bounded}) \quad 0 \quad (\text{bounded}) \quad 3$$

So for small x on either side of 0, $f'(x)$ has the same sign as $3x^2$, namely positive.

3. (Proof that f'' changes sign infinitely often in any neighborhood of $x = 0$):

$$f''(x) = 240x^3\sin(1/x) + 60x^4\cos(1/x) \cdot (-1/x^2) - 36x^2\cos(1/x)$$

$$+ 12x^3\sin(1/x) \cdot (-1/x^2) + 6x$$

$$= 240x^3\sin(1/x) - 96x^2\cos(1/x) + 6x - 12x\sin(1/x)$$

$$= x\left[\left(240x^2\sin(1/x) - 96x\cos(1/x)\right) + 6\left(1 - 2\sin(1/x)\right)\right]$$

$$\downarrow \quad \Downarrow \quad \downarrow \quad \Downarrow \quad \Downarrow$$

$$0 \quad (\text{bounded}) \quad 0 \quad (\text{bounded}) \quad (\text{oscillating between } \pm 2)$$

So for small x on either side of 0, $f''(x)$ has the same sign as $18x$ infinitely often and the same sign as $-6x$ infinitely often. There can be no well-defined change in concavity at $x = 0$, and so the graph does not have a point of inflection there.

A simpler function with the same property (brought to my attention by Ray Cannon) is $f(x) = x^5 + x^6\sin(1/x)$. It requires a little more trigonometry to show that f'' changes sign infinitely often in every neighborhood of 0, but the interested reader can verify that it does.

Incidentally, as you might have suspected by now, the student's argument is actually valid if it is known that the function only has a finite number of inflection points near the critical point in question. (The proof, omitted by the student, is a nice exercise.)

4. When Is A Maclaurin Series Not A Maclaurin Series?

BC-2 in 1996 was about Maclaurin series:

2. The Maclaurin series for $f(x)$ is given by $1+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\cdots+\frac{x^n}{(n+1)!}+\cdots$
   (a) Find $f'(0)$ and $f^{(17)}(0)$.
   (b) For what values of x does the given series converge? Show your reasoning.
   (c) Let $g(x)=xf(x)$. Write the Maclaurin series for $g(x)$, showing the first three nonzero terms and the general term.
   (d) Write $g(x)$ in terms of a familiar function without using series. Then, write $f(x)$ in terms of the same familiar function.

I had no problem recognizing this as a "series manipulation" problem when I took the exam in the privacy of my own room. The series in part (c) was clearly the series for $e^x - 1$, and so my answer in part (d) was:

$$g(x) = e^x - 1;$$
$$f(x) = \frac{e^x - 1}{x}.$$

When I arrived at the reading I discovered that I had only scored 8 out of 9 on BC-2. I am not sure whether the Committee had intended to catch me on a technicality, but I had certainly taken their bait, including the hook, line, and sinker. It was somewhat consoling to note, after all the exams had been graded, that I was not the only sucker in the Committee's boat: flopping around beside me were more than 99% of the BC population. (There were only 5 perfect solutions among the 21,020 BC exams that year.) The problem, you see, is that $f(x)=\frac{e^x-1}{x}$ is not even *defined* at $x=0$, let alone infinitely differentiable, and is therefore not eligible to have a Maclaurin series. The correct f should be

$$f(x) = \begin{cases} \dfrac{e^x - 1}{x} & \text{for } x \neq 0, \\ 1 & \text{for } x = 0. \end{cases}$$

For additional perspective let me hearken back to 1993, when the following problem appeared as BC-5:

5. Let f be the function given by $f(x)=e^{\frac{x}{2}}$.
   (a) Write the first four nonzero terms and the general term for the Taylor series expansion of $f(x)$.
   (b) Use the result from part (a) to write the first three nonzero terms and the general term of the series expansion about $x=0$ for $g(x)=\frac{e^{\frac{x}{2}}-1}{x}$.
   (c) For the function g in part (b), find $g'(2)$ and use it to show that $\sum_{n=1}^{\infty}\frac{n}{4(n+1)!}=\frac{1}{4}$.

Notice in part (b) the clever avoidance of the names "Taylor" and "Maclaurin." This allows the student to construct a power series of the form $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$ without being concerned about whether or not $g(0)$, $g'(0)$, $g''(0)$, etc., exist—which they do not.

Since learning my lesson in 1996 I have enjoyed looking through various textbooks to see how many include exercises in their "series manipulation" sections that look like this:

Use the Maclaurin series for $\cos x$ to construct a Maclaurin series for $\frac{\cos x - 1}{x}$.

Each one I find means that there is one more sucker in the boat.

5. The Differential Equation with Too Many Solutions

This was BC-6 in 1993:

6. Let f be a function that is differentiable throughout its domain and that has the following properties.
   (i) $f(x+y) = \frac{f(x)+f(y)}{1-f(x)f(y)}$ for all real numbers x, y, and $x+y$ in the domain of f.
   (ii) $\lim_{h\to 0} f(h) = 0$
   (iii) $\lim_{h\to 0} \frac{f(h)}{h} = 1$
   (a) Show that $f(0) = 0$.
   (b) Use the definition of the derivative to show that $f'(x) = 1 + [f(x)]^2$. Indicate clearly where properties (i), (ii), and (iii) are used.
   (c) Find $f(x)$ by solving the differential equation in part (b).

Not only was I chair of the Committee when this problem appeared, I actually wrote the thing. It then went through the usual rigorous scrutiny and polishing, wherein quite a few competent people solved it multiple times. All this quality control notwithstanding, it was only at the reading itself that someone (the Chief Reader) discovered that there was a lot more involved in pinning f down than anyone had theretofore realized (including the author of the problem).

The solution to the differential equation is pretty straightforward: You separate variables and find that $f(x) = \tan(x + C)$. One is then supposed to use the initial condition from part (a) to conclude that $\tan(C) = 0$, from which it follows that $f(x) = \tan(x + k\pi) = \tan x$. It looks simple enough, but it is not that simple. Can you see why?

Here is the rub: the initial condition is fine for "pinning down" the function at $x = 0$, but the function we are talking about is the *tangent* function, which has asymptotes. So we can narrow our solution down to $y = \tan x$ on the interval $(-\pi/2, \pi/2)$, but *outside* that interval we could jump to a different, shifted tan curve without affecting $f(0)$ a bit! Fortunately for the integrity of the exam, condition (i) precludes this from happening. (This is another nice exercise.) Still, nobody expected the BC students to catch this subtlety, let alone prove their way out of it, so the grading standard was written without mentioning it. It was those of us who discussed the subtlety at the grading who gained some wonderful insights into differential equations.

6. Methods of Solution That Should Not Work

Few of us will ever forget the "cola problem" on the 1996 exam:

3. The rate of consumption of cola in the United States is given by $S(t) = Ce^{kt}$, where S is measured in billions of gallons per year and t is measured in years

from the beginning of 1980.

(a) The consumption rate doubles every 5 years and the consumption rate at the beginning of 1980 was 6 billion gallons per year. Find C and k.

(b) Find the average rate of consumption of cola over the 10-year period beginning January 1, 1983. Indicate units of measure.

(c) Use the trapezoidal rule with four equal subdivisions to estimate $\int_5^7 S(t)\,dt$.

(d) Using correct units, explain the meaning of $\int_5^7 S(t)\,dt$ in terms of cola consumption.

There were numerous lessons we learned from this problem, including the influence of units of measurement on the constants C and k. The most unexpected lesson, however, came (as usual) from a student solution.

Here is the way you expect students to find the average value in part (b):

$$\text{Average value} = \frac{1}{b-a}\int_a^b Ce^{kt}\,dt = \left(\frac{1}{b-a}\right)\left(\frac{C}{k}\right)\left(e^{kb} - e^{ka}\right).$$

A few students, however, proceeded quite differently. They found the "special c" value guaranteed by the Mean Value Theorem for derivatives (which I will call p here to avoid confusion with C), then plugged it into the function S:

$$S'(p) = kCe^{kp} = \frac{Ce^{kb} - Ce^{ka}}{b-a}$$

$$\Rightarrow S(p) = Ce^{kp} = \left(\frac{1}{k}\right)\frac{Ce^{kb} - Ce^{ka}}{b-a}.$$

Notice that the answers are the same! This is no coincidence arising from the numbers involved, either; that is why I used all those general constants. It is actually *true*, for an exponential function, that the average value of the function occurs at the point found by the MVT for derivatives! Is this a general theorem?

Well, of course not. Consider the graph of $y = \sin x$ on the interval $[0, \pi]$, for example (Figure 2). The MVT value occurs where the derivative is zero, and the y-coordinate there is the maximum, not the average, value.

It is not even true for monotonic functions in general (as the curious reader is invited to check), but since it is true for exponential functions, the students got full credit!

1. Given the graph of $y = f'(x)$ at the right, explain why the graph of $y = f(x)$ has a point of inflection at $x = 1$.

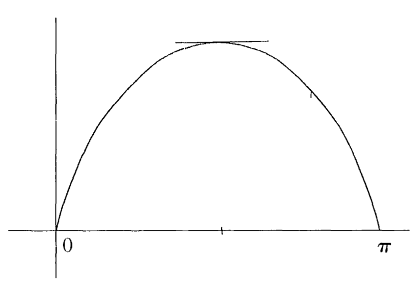


Figure 2

And that wasn't the only surprise on the 1996 test. Check out AB-2:

2. Let R be the region in the first quadrant under the graph of $y = 1/\sqrt{x}$ for $4 \le x \le 9$.
   (a) Find the area of R.
   (b) If the line $x = k$ divides the region R into two regions of equal area, what is the value of k?
   (c) Find the volume of the solid whose base is the region R and whose cross sections cut by planes perpendicular to the x-axis are squares.

Here is a reasonable approach to solving part (b):

$$\int_a^k \frac{1}{\sqrt{x}}\,dx = \int_k^b \frac{1}{\sqrt{x}}\,dx$$

$$2\sqrt{x}\,\Big|_a^k = 2\sqrt{x}\,\Big|_k^b$$

$$2\sqrt{k} - 2\sqrt{a} = 2\sqrt{b} - 2\sqrt{k}$$

$$4\sqrt{k} = 2\sqrt{a} + 2\sqrt{b}$$

$$\sqrt{k} = \frac{\sqrt{a} + \sqrt{b}}{2}.$$

For our interval $[4, 9]$ we conclude that $\sqrt{k} = \frac{5}{2}$, and so $k = \frac{25}{4}$.

Again, I left all the letters in the solution so that I could contrast it with another method of solution that was actually used by some students. These creative individuals chose to find k by finding the x in $[4, 9]$ at which the average value of the function occurs. (This is the number whose existence is guaranteed by the Mean Value Theorem for Integrals.) Here's the work:

$$\frac{1}{\sqrt{k}} = \frac{1}{b-a}\int_a^b \frac{1}{\sqrt{x}}\,dx$$

$$\frac{1}{\sqrt{k}} = \frac{1}{b-a}\left(2\sqrt{x}\,\Big|_a^b\right)$$

$$\frac{1}{\sqrt{k}} = \frac{2(\sqrt{b} - \sqrt{a})}{b-a}$$

$$\frac{1}{\sqrt{k}} = \frac{2}{\sqrt{b} + \sqrt{a}}$$

$$\sqrt{k} = \frac{\sqrt{a} + \sqrt{b}}{2}$$

Needless to say, these students got the same answer. In fact, they also got the same credit, since we can see that their algorithm is valid for this function. But is this a general theorem?

Once again, the answer is, "Of course not." Indeed, the same picture of $y = \sin x$ on $[0, \pi]$ that debunked the previous "theorem" will serve to debunk this one, for virtually the same reason: the area splits evenly where the function value is the maximum, not the average, value of f on the interval.

So, in these two problems did the students who used the "methods that should not work" actually know what they were doing? In all likelihood, no. Did they really deserve full credit? According to our rubrics, yes. Notice that these students *showed their work* and demonstrated that they were using what was, like it or not, a *valid algorithm for solving the given problem.* This is quite a different story from those frequently-seen papers that have correct answers with little or no work. In those cases we take off credit precisely because we *cannot tell* whether the students are using a valid algorithm or not. We can take off credit for mathematics that is not there, but as the current Chief Reader, Bernie Madison, has often pointed out, We can't take off credit for correct mathematics.

On that philosophical quote I will end my brief historical tour of the Lessons I have Learned from AP Readings Past. If you have never had the experience of grading exams all day every day for a week, I urge you to become involved as an AP reader. Then, for your continued happiness and professional development, I wish that each of you might enjoy learning experiences similar to these examples at every reading you attend. Meanwhile, I hope that this paper might have given you a few in absentia.

Compare and Contrast

By canceling the 6s, we see that

$$\frac{16}{64} = \frac{1}{4}.$$

By canceling the 3s, we see that

$$\frac{5^3 + 2^3}{5^3 + 3^3} = \frac{7}{8}.$$

Both equations are correct for the same incorrect reason. The first, though, is an accident of the decimal system, while the second has universality:

$$\frac{(a+b)^3 + a^3}{(a+b)^3 + b^3} = \frac{2a+b}{a+2b}$$

is an identity. Are there any others that are similar?