How Advanced Calculus helped me become a better high school teacher

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Introduction:

I attended the New York component of the PMET Workshops for Secondary Education in the summers of 2003 and 2004. Since I am trained as a mathematician, these workshops exposed me to a variety of new ideas and prodded me to rethink my approach to teaching mathematics courses intended for high school teachers.

One such course is Advanced Calculus. In the fall of 2004, Emporia State offered it as an online course as part of a masters program for in-service teachers. Advanced Calculus is typically difficult and sometimes accused of being far-removed from high school mathematics. As the instructor of the course, my goal was to use the ideas presented in the PMET workshops to enhance the course and make it relevant for teachers. As the MET report states, "… most [math students] acquire only procedural facility in using formulas involving functions for calculations, not a deep understanding of functions and related concepts like limits or continuity. Thus, it is important for prospective teachers to revisit the elementary functions of high school mathematics from an advanced standpoint." [1, p. 43]

As I reflected on the various discussions from the PMET workshops, four main themes developed. A math teacher should:

- have a "mathematician's perspective,"
- understand the fundamental principles of mathematics,
- see the relevance and place of individual topics within a larger context, and
- be able to apply his or her knowledge in the classroom.

These were the points that I wanted to emphasize throughout the semester, beginning with a questionnaire for the first class period. About a year after the course was finished, I contacted some of the students in the class. I wanted to see if they found any of the topics we discussed in class beneficial to them as teachers. I've included some of their responses in the discussion that follows.

Developing a "mathematician's perspective":

A recurring point is that prospective teachers should move beyond viewing math as a set of algorithms to be applied in various situations. They should see the elegance of a well-constructed proof and the abstract beauty that mathematics can hold. To see the students' general views of math, the following question was asked on the first day:

"The mathematics department is in the College of Arts and Sciences. So is math in the arts or sciences?"

I was hoping that the students would share the perspective that mathematics is as much of an art as it is a science. Fortunately, some students had already made that observation. Here were some of the responses:

- “Math is a science because it is practical and useful.”
- “Math is more scientific in my mind because it operates under a set of rules and guidelines whereas art implies more of an ‘un-ruled’ discipline. Although, many times writing these proofs feels very ‘un-ruled’ too!”
- “There is beauty in a concise, articulate, and accurate proof.”
- “Mathematics is both an art and a science. Mathematics is an art because writing clear, precise proofs takes creativity. Also, the product of that creative process can be improved continually. Mathematics is a science because there are established truths and methods that are used for moving from one point to another.”

Due to the rigorous nature of Advanced Calculus, there are many occasions for students to see clever and unique proofs. But there are other opportunities to foster a mathematical outlook. For instance, one student particularly enjoyed learning the method of Archimedes to use sequences to calculate π. Also, learning about the Sequential Characterization of Limits helped students realize the benefit of approaching a problem from an alternative view. In this case, proving a result about functions was more easily accomplished by viewing the situation in terms of sequences, and then using known results about sequences to more easily finish the proof. About this topic, one teacher later remarked, "This was useful because I try to teach students to see things from different viewpoints. It is always useful to try to do what you want your students to do."

Understanding the fundamental principles:

Another theme evident from the PMET workshops is the idea that students should be aware of the mathematical underpinnings of a subject. They should know the difference between axioms, definitions, and theorems. They should also learn how axioms and definitions have developed and changed over time. To emphasize some of these ideas on the first day of class, I asked the following question:

“Why is 0<1? Is it a definition, an axiom, a theorem...?”

Here were a few responses:

- “Because zero represents nothing and one is a number that means there is something.”
- “Zero is less than one because that's how we have defined those two numbers.”
- “It is defined that way, but also could probably be proven.”
- “...I don't think it can really be explained. 0<1 because it is.”
- “...It's a basic axiom.”
- “...because 0 is to the left of 1 on the number line.”

The responses exhibit a range of ideas, some more clear than others. A point of discussion was that we have to understand what we mean by 0, 1, and ‘<’ before proceeding. This leads to a formulation of the field and order axioms where we assume 0 and 1 are the additive and multiplicative identities, respectively. Then 0<1 can be proven.

From this point, we can see how the rest of the semester's results will be built from these various axioms. About this, one teacher later remarked, "I feel that this understanding has made me a better teacher because I am not just teaching things in isolation but as parts of a whole."

Advanced Calculus also gives the opportunity to demonstrate the importance of precise definitions. We are able to discuss historical examples of the pitfalls associated with a vague understanding of an idea such as continuity or a limit. I was encouraged to hear how these ideas transfer to the high school classroom. As one teacher commented, "Someone will wonder why it [a definition] has to be so specific and 'complicated,' and then I get to talk about what would happen if the definition was not so well-defined."

The role of proof permeates the entire course as well. One teacher summed it up in this way, "I typically try to derive many of the theorems so they can see that I am not making things up out of thin air." Another teacher described the increased credibility it gave him in the classroom. When a student asked to see a proof of a theorem in a calculus course, he had the confidence and ability to provide the proof.

Another area of emphasis is the concept of function. The MET report recommends: "Prospective high school mathematics teachers need to acquire deep understanding of the concept of a function and of the most important classes of functions (polynomials, exponential and logarithmic, rational, and periodic)." [1, p. 42] An example of this is seeing how the logarithm and exponential functions are defined. The logarithm is defined using an integral, and then students can use properties of integration to derive the properties of logarithms. Similarly, we prove that the exponential function is the only function continuous at $x = 0$ that satisfies $f(x + y) = f(x)\,f(y)$. These topics expand the teachers' knowledge, while reinforcing the basic concepts that they will be communicating to their own students.

Seeing the relevance and place of individual topics within a larger context:

Another theme from the PMET workshops is that future teachers should be able to see the connections between different areas of math and see the "big picture." We have already seen examples of this when dealing with axioms, definitions and so on. As a more concrete example, the first day I asked the following (well-known) question:

"Is 0.999...=1? Why or why not?"

Some responses included:

- "Definitely...every calculator makes it that way."
- "No, just because a number is infinitesimally close to another number does not make it the same value."
- "Yes, divide both sides by 3."
- "No, because each number is unique. There is a one-to-one mapping of the real numbers."
- "I would think not. I've seen a 'proof' to suggest it. But like the famed 1=2 proof, I'm sure there's a logic error that I can't recall."
- "0.999... = 0.9/(1 - 0.1) = 0.9/0.9 = 1."

It is interesting that some of the responses employ appropriate terminology, such as "infinitesimally close," but arrive at the wrong conclusion. As the last response alludes to, one must understand the definition of a decimal expansion and the ideas of the infinite, limits, and series to arrive at a correct answer. One teacher later remarked, "This was pretty cool. I am even going to teach the basic idea to my Algebra 2 students later this week. I'm not sure how the idea escaped me in my previous education, but it did."

One benefit of seeing how different ideas relate is that it leads to better understanding of the individual topics. For instance, limits help us to know why and when rational functions have asymptotes. One teacher had this to say, "This is another area that we discuss in Alg. 2 and PreCalc. My students...wanted rules to follow. I keep plugging away at the limit idea however, because I think understanding beats rote memorization of rules." Another example is the relationship between the exponential and logarithmic functions as inverses. This observation then allows us to prove various properties of those functions. A teacher observed, "I do try to emphasize that the logarithm and exponential functions are inverses of one another.... Other teachers in my building just teach the rules. I like to at least show students how they are all related to one another."

Applying knowledge in the classroom:

The most evident need for prospective teachers is to acquire knowledge that they can apply directly in the classroom. In fact, this one area of teacher preparation is probably overemphasized. The PMET workshops reiterated the point that a teacher's education should encompass more than just learning the specific skills that will be taught to high school students. Nonetheless, these skills and ideas are important as well. On the first day, I posed the following question that is relevant in most high school math classes:

"What is a real number?"

Students responded with:

- "A number which has meaning in application in the real world."
- "Any integer, fraction, decimal I can think of as long as it doesn't contain the square root of a negative number."
- "Any number that can be expressed on the real number line."
- "The set of all numbers, which includes rational and irrational numbers."
- "A number that is in the set of numbers satisfying the field and order axioms."

Although the answers above are not entirely incorrect, most indicate either circular or incomplete reasoning. Early in the semester we were able to clarify this answer by assuming the real numbers are characterized by a certain set of axioms. One teacher found this helpful because "I spend time on axioms at the beginning of Alg 2 and Pre-Calc...also Geometry."

It is worth noting that not everything we teach to future teachers will be used in the classroom, even if it helps them become more effective overall. After discussing the underlying axioms, we moved on to the density of the rationals and irrationals and the uncountability of the reals. We discussed some of the apparent contradictions involved. One teacher commented, "I guess that I am still pondering the wonder of it all. Sometimes I make comments about it to students, but they don't really seem to care to think about it."

As more high school teachers are expected to teach Calculus, the course becomes more relevant in practical ways. For example, we discussed the definition of a definite integral using the more general notation, $||P||\to 0$. One of the teachers later mentioned that this notation was also used in the text he was using in his own class. He was pleased to be familiar with the notation.

Conclusion:

For a class that is sometimes associated only with ε-δ proofs, I was pleased to see the variety of connections the teachers made with the course and their own experiences as teachers. As one teacher explained, "I don't necessarily expect to teach all of this, or that all of my students understand it all, but I sure want to give them the chance to see that it should make sense, that there is a reason for the definitions, and I think it is very important that they see that it isn't something randomly that we come up with."

Another positive result was the increased confidence and credibility for the teacher in the classroom. Another teacher described it in this way:

"A good understanding of the concepts also comes in really handy during question and answer time. Invariably someone will ask a question that is deeper than they know or that is a bridge to them having deeper understanding, the teacher's ability to answer those questions well on the spot can give credence to not only their subject awareness, but also the study of the subject material."

It is worth mentioning that these observations come from just a small subset of the teachers from the Advanced Calculus course. It is unlikely that all of them would share all these views. However, it is encouraging that these points were made from experience in the high school classroom, rather than my own opinions. In the future, I believe it would be helpful for me to encourage my students to reflect on these themes throughout the semester. I also want the students to have a broader view of what a college math class should mean to them. Although it is important to take specific ideas back to the classroom, I want to emphasize the benefit of examining "simple" concepts more deeply and view mathematics for the elegant subject that it is.

References:

[1] Conference Board of the Mathematical Sciences. (2001) The Mathematical Education of Teachers in *Issues in Mathematics Education* (11). Providence, RI: American Mathematical Society.