

Computer-Aided Delusions

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The College Mathematics Journal, September 1993, Volume 24, Number 4, pp. 366–369

The so-called real numbers used in computing are actually a small subset of the rational numbers. This simple fact can easily lead to some interesting effects. The example we study in this note illustrates a pitfall that may be encountered when studying the graph of a function with the aid of a computer. Demana and Waits [1] discuss some interesting examples in which computer generated graphs overstep important features of a function or suggest erroneous asymptotic values. The example we discuss here illustrates a case where a computer-generated graph appears to indicate incorrect slope behavior at a point.

We consider the following well-known continuous function used in the teaching of calculus to illustrate logarithmic differentiation:

$$f(x) = \begin{cases} |x|^x, & x \neq 0 \\ 1, & x = 0. \end{cases} \quad (1)$$

Suppose that we were interested in the shape of f and, in particular, suppose that we were curious about the slope S_0 of $f(x)$ at $x = 0$. Figures 1 and 2 indicate a slope of

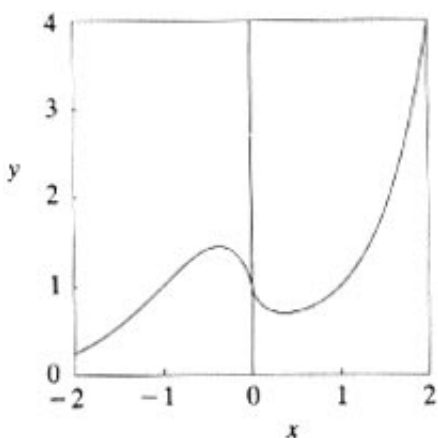


Figure 1

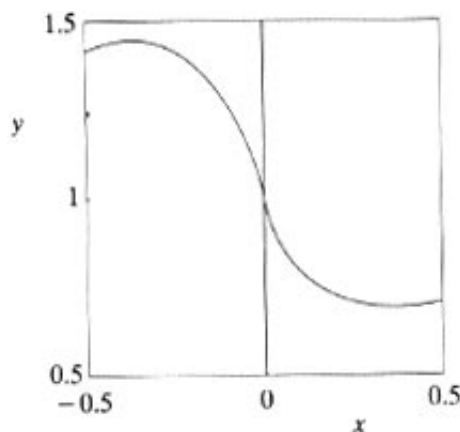


Figure 2

about $S_0 \approx -4.5$, as estimated by laying a ruler along the graph and measuring its slope. Since the graphs are very nice and they were, after all, generated by a *computer*, this rough estimate might be considered satisfactory.

However, imagine we were to look a little more closely at the problem, and plot the graphs shown in Figures 3 and 4, which show magnified portions of the graph of f near $x = 0$. Figure 3 indicates a slope of about $S_0 \approx -9$, and Figure 4 indicates a slope of about $S_0 \approx -18$. Similar graphs covering x intervals with half-widths of respectively 10^{-10} and 10^{-20} indicate slopes of about -27 and -50 . This really is turning out to be a slippery slope.

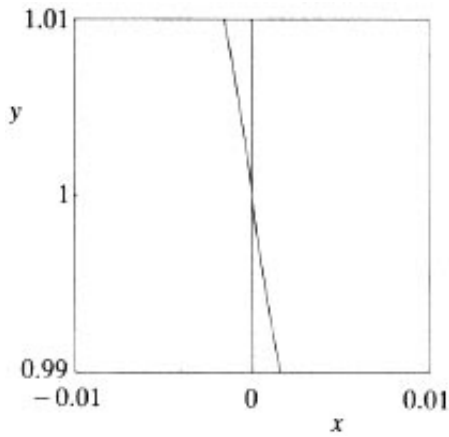


Figure 3

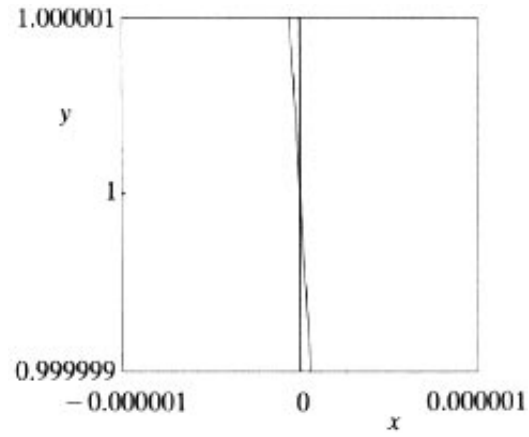


Figure 4

Since the function f is elementary, we can analyze the situation exactly. Using logarithmic differentiation on (1) we find

$$f'(x) = f(x)(\ln|x| + 1). \quad (2)$$

Consequently, since $f(x)$ has limit 1 as x tends to 0, we see that the slope is unbounded in this limit; evidently $f'(x)$ tends to $-\infty$. The approach to large (negative) values is *very* slow because of the nature of the log function. Some approximate values of $f'(x)$ are shown in Table 1. In Figure 5 we exhibit graphs of a symmetrical approximation

$$g(x, h) = \frac{1}{2h} [f(x + h) - f(x - h)]$$

for the slope of f for values of h diminishing from $h = 0.1$ down to $h = 0.01$ in steps of size 0.01. As h decreases, the minimum of $g(x, h)$ with respect to x decreases without bound.

Table 1

n	$f'(10^{-n})$	n	$f'(10^{-n})$
1	-1.03	10	-22.03
2	-3.44	20	-45.05
3	-5.87	30	-68.08
4	-8.20	40	-91.10
5	-10.51	50	-114.13
6	-12.82	60	-137.16
7	-15.12	70	-160.18
8	-17.42	80	-183.21
9	-19.72	90	-206.23
10	-22.03	100	-229.26

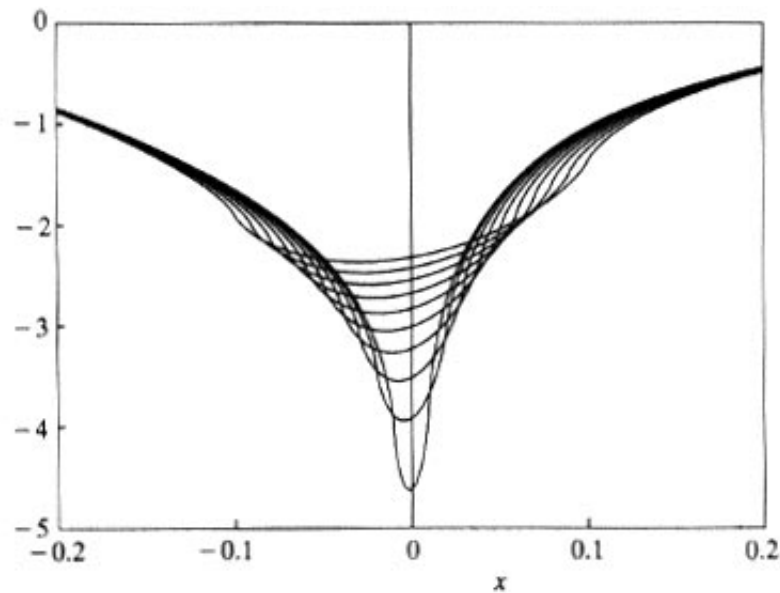


Figure 5

Of course, alternatively, we could use a computer to investigate the slope S_0 without graphics; we could simply compute values of $f(x)$ near $x = 0$. If none of the points we consider is *very* close to 0, then it is clear in principle that the same misleading evidence concerning S_0 could arise from a purely numerical investigation.

The pitfall we address is not to do with accuracy nor with computer arithmetic. The danger arises because the graphical presentation of the data, with straight-line segments joining adjacent computed points, may easily *suggest* to us that we have explored the situation in enough detail when in fact we have not. The graphs of f shown in Figures 1 and 2 look extremely convincing. In fact, they are just the kind of computer-generated graphs we should be proud to include in a research report. We usually turn to the computer, not for classroom examples like (1) but when the going gets tough, then even computing *values* of the function f is not easy. This, of course, is precisely the situation where we do *not* have analytical expressions like (2) available to us for checking our results. The lesson is a very general one. We should always be very careful to explore the territory round about any result that we first find with the aid of a computer. Perhaps we should remember one of Alan Turing's exhortations made towards the beginning of the present era of computing with the aid of electronic machines, that *one should be fair to computers*.

Acknowledgement. The author gratefully acknowledges partial financial support of this work by the Natural Sciences and Engineering Research Council of Canada.

Reference

1. F. Demana and B. K. Waits, Pitfalls in graphical computation, or Why a single graph isn't enough, *College Mathematics Journal* 19 (1988) 177-183.