Designing a Rose Cutter

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Most students of mathematics appreciate demonstrations of the relevance of mathematics in their chosen fields of study, and engineering students essentially insist on them. While almost all ordinary differential equations books include a brief discussion of first-order differential equations of the form $y' = g(y/x)$, few provide an example from engineering. I offer one here.

The problem is to design blades for a pair of pruning shears, consisting of one straight blade and one curved blade, with the specification that the angle between the two blades be constant regardless of how far the jaws are open.

Figure 1 shows the blades in the open position. The edge of the straight blade is segment OB, with the hinge point at the origin. We assume that $\theta_0 = \pi/3$ radians and that each blade measures 5 cm from the hinge point to the tip, so $|OA| = |OB| = 5$. Now we fix

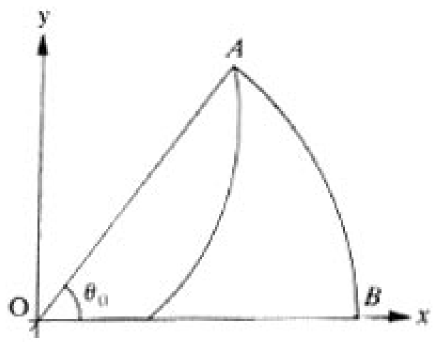

Figure 1

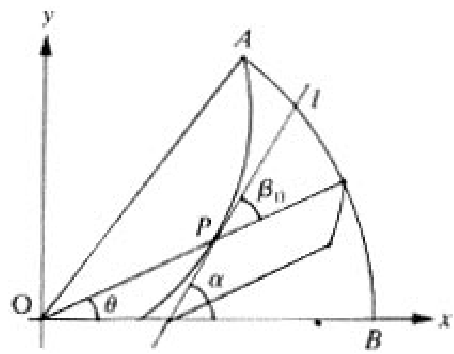

Figure 2

the curved blade in the open position and allow the tip of the straight blade to move along the circular arc AB as the jaws close. Figure 2 shows the straight blade in an arbitrary intermediate position. The design specification requires that β_0 be constant. For convenience, we select $\beta_0 = \pi/4$. Let the function representing the curved blade be $y = f(x)$ and the point P of the intersection of the blades be (x, y).

The slope of the tangent line l can be expressed as $dy/dx = \tan(\alpha)$. But $\alpha = \beta_0 + \theta$, so

$$\tan(\alpha) = \tan\left(\frac{\pi}{4} + \theta\right) = \frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4}\tan\theta}.$$

Because $\tan\theta = y/x$, we have the differential equation

$$\frac{dy}{dx} = \frac{1 + \dfrac{y}{x}}{1 - \dfrac{y}{x}}$$

as a model for the curved blade edge.

This differential equation becomes separable by our introducing the new dependent variable $v = y/x$, which implies that $y' = v + v'x$. Thus, the differential equation becomes

$$v + v'x = \frac{1 + v}{1 - v}, \qquad \text{or} \qquad \frac{1 - v}{1 + v^2}\,dv = \frac{1}{x}\,dx.$$

Integration yields

$$\text{Arctan}\,(v) - \frac{1}{2}\,\text{In}\,(1 + v^2) = \text{In}\,|x| + C, \qquad \text{or}$$

$$\text{Arctan}\left(\frac{y}{x}\right) - \frac{1}{2}\,\text{In}\left(1 + \frac{y^2}{x^2}\right) = \text{In}\,|x| + C.$$

As often happens with separable equations, this implicit general solution does not yield a solution in the form $y = f(x)$, so does not lend itself to recognition of the shape of the

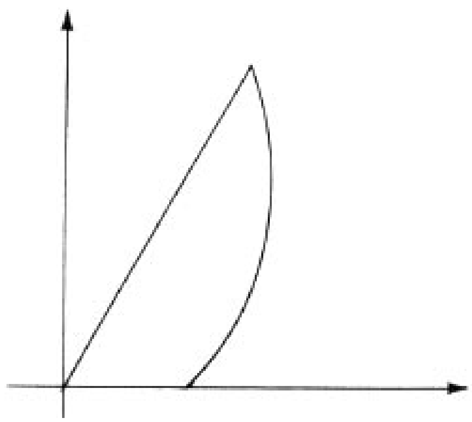

Figure 3

curved blade. But further simplification yields $\text{Arctan}\,(y/x) - \ln\sqrt{x^2 + y^2} = C$, which suggests. . . polar coordinates! The polar form of the general solution is the logarithmic spiral $\theta - \ln r = C$ or $r = ke^{\theta}$.

Since the polar point $(5, \pi/3)$ is on the curve, $k \cong 1.8$ and the solution to our design problem is

$$r = 1.8e^{\theta} \quad \text{for } 0 \leq \theta \leq \pi/3.$$

Figure 3 shows a plot of the actual blade shape, which could be scaled to build a template for the blade manufacturing process.

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