What Goes Up Must Come Down; Will Air Resistance Make It Return Sooner, or Later?

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A ball thrown straight up with speed v_i would, in the absence of air, return in time $2v_i/g$. Air resistance, or drag, will influence the return time in two ways: the maximum height reached is less than the zero-drag height $v_i^2/2g$, and the speed at any height z is less than the zero-drag speed. (These statements follow from the energy equation $\frac{1}{2}mv_i^2 = \frac{1}{2}mv(z)^2 + mgz + W$, where m is the mass of the ball, and W is the (positive) work done against air resistance. The speed is zero at the top of the trajectory, so $z_{\text{max}} < v_i^2/2g$; and at any z, $v(z) < \sqrt{v_i^2 - 2gz}$. Note that the energy conservation equation is not an additional physical principle: it follows from the equation of motion on multiplying by v and integrating.) Thus with air resistance, the ball has a *shorter distance* to travel, but at a *slower speed.* What effect wins?

Let $f(v)$ be the deceleration due to the drag force. The equation of motion then reads $dv/dt = -g - f(v)$ on the way up, and $dv/dt = g - f(v)$ on the way down (it is convenient to deal with speeds rather than velocities in this context). We will assume that $f(v)$ has the property that there is just one speed at which the gravitational and drag forces are in balance. This defines the *terminal speed* v_t: $f(v_t) = g$. The terminal speed is a natural scaling parameter for this problem. Let $u = v/v_t$ and $\phi(u) = f(v)/f(v_t) = f(v)/g$. Then by integrating dt (obtained from the equation of motion) we find the time to go up to maximum height is

$$t_{\text{up}} = \int_0^{v_i} \frac{dv}{g + f(v)} = \frac{v_t}{g}\int_0^{u_i} \frac{du}{1 + \phi(u)}, \tag{1}$$

and the time to come down is

$$t_{\text{down}} = \int_0^{v_f} \frac{dv}{g - f(v)} = \frac{v_t}{g}\int_0^{u_f} \frac{du}{1 - \phi(u)}. \tag{2}$$

The speed on impact, v_f, is determined by the condition that the distance travelled on the way up is the same as that travelled on the way down. These are given by integrating $v\,dt$; we find u_f is determined by

$$\int_0^{u_i} \frac{u\,du}{1 + \phi(u)} = \int_0^{u_f} \frac{u\,du}{1 - \phi(u)}. \tag{3}$$

We are interested in the ratio τ of the return time to the zero-drag return time $2v_i/g$. From (1) and (2),

$$\tau = \frac{t_{\text{up}} + t_{\text{down}}}{2v_i/g} = \frac{1}{2u_i}\left[\int_0^{u_i} \frac{du}{1 + \phi} + \int_0^{u_f} \frac{du}{1 - \phi}\right]. \tag{4}$$

Physically, $f(v)$ must go to zero as v goes to zero. Thus Φ, the maximum value of $\phi(u)$, can be made arbitrarily small compared to unity when the initial speed v_i is chosen sufficiently small compared to the terminal speed v_t (u_i sufficiently small). We can therefore expand $[1 \pm \phi(u)]^{-1}$ in (3) and (4), to find

$$\frac{u_f}{u_i} = 1 - \frac{2}{u_i^2}\int_0^{u_i} u\phi\, du + O(\Phi^2)$$

$$\tau = 1 - \frac{1}{{u_i}^2}\int_0^{u_i} u\phi\, du + O(\Phi^2). \tag{5}$$

Thus any physically reasonable form of drag will make the ball return sooner, provided the launch speed is small compared to the terminal speed.

Wind tunnel experiments [**1**] on spheres show that the drag force is (approximately) proportional to v^2 in the Reynolds number range $10^3 \le R \le 10^5$. This covers the range of practical interest, provided the launch speeds are kept moderate (a sphere of diameter 1.5 cm and speed 10^3 cm/s has $R \cong 10^4$ in air). For $f = kv^2 (\phi = u^2)$ we find from (3) and (4) that

$$u_f = \frac{u_i}{\sqrt{1 + u_i^2}} \tag{6}$$

and

$$\tau = (\arctan u_i + \operatorname{arctanh} u_f)/2u_i. \tag{7}$$

The numerator $N(u) = \arctan u + \operatorname{arctanh}\left(u/\sqrt{1 + u^2}\right)$ has slope $dN/du = (1 + u^2)^{-1} + (1 + u^2)^{-1/2}$, which is less than 2 for nonzero u. Thus $N(u_i)$ increases more slowly than $2u_i$, the leading term in its Taylor expansion about $u_i = 0$. It follows that, for a v^2 drag, τ is always less than unity, no matter what the initial speed.

Could this result be true for an arbitrary (nonnegative) drag $f(v)$? Let's try a few more examples. When f is linear in v (Stokes' law), we find the attractive result

$$\tau = \frac{1}{2}\left(1 + \frac{u_f}{u_i}\right)$$

or

$$t_{\text{up}} + t_{\text{down}} = \frac{v_i + v_f}{g}. \tag{8}$$

Since v_f is always less than v_i, we again have the return time being shortened by air resistance, irrespective of the initial speed.

So far, all has indicated a shorter return time. Now consider some fractional powers. First suppose $f(v) \sim v^{1/2}$. Setting $u = w^2$, we find that u_f is determined by an interesting transcendental equation

$$\frac{1}{3}{w_i}^3 - \frac{1}{2}{w_i}^2 + w_i - \log(1 + w_i) = -\frac{1}{3}w_f^3 - \frac{1}{2}w_f^2 - w_f - \log(1 - w_f), \tag{9}$$

and that the ratio of return time to zero-drag return time is

$$\tau = {w_i}^{-2}\{w_i - \log(1 + w_i) - w_f - \log(1 - w_f)\}. \tag{10}$$

For $u_i \gg 1$ we find $\tau \to \frac{1}{3} u_i^{1/2}$, larger than unity.

Next, suppose $f(v) \sim v^{2/3}$. Setting $u = y^3$ we find

$$\frac{1}{2} y_i^4 - y_i^2 + \log(1 + y_i^2) = -\frac{1}{2} y_f^4 - y_f^2 - \log(1 - y_f^2) \tag{11}$$

and

$$\tau = \frac{3}{2y_i^3}\{y_i - \arctan y_i + \operatorname{arctanh} y_f - y_f\}. \tag{12}$$

For $u_i \gg 1$, $\tau \to \frac{3}{8} u_i^{1/3}$, again larger than unity.

The above results suggest to me that there is a cross-over at the linear force law:

CONJECTURE. *For powers p in $f(v) = kv^p$, $p \geq 1$ gives a return time which is always shorter than the zero-drag return time $2v_i/g$. For $p < 1$, the return time is shorter for small initial speeds, but eventually becomes longer than $2v_i/g$ as v_i increases. The closer p is to 1, the higher the ratio of the initial speed to the terminal speed before this happens.*

We have determined $\tau(u_i)$ for only four values of p: 2, 1, 1/2, 2/3. Students may enjoy some of the following projects in analysis and numerical methods:

(a) plotting τ versus u_i for these four values of p;

(b) finding other values of p for which the integral equation (3) for u_f is reducible to a transcendental equation, and plotting $\tau(u_i)$ for these;

(c) a class exercise in which different values of $p < 1$ are assigned to students or student groups, and each is asked to find the u_i for which $\tau = 1$.

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Reference

[**1**] G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge, 1967, p. 341.