

The Disk and Shell Method

Charles A. Cable

Department of Mathematics, Allegheny College, Meadville, PA 16335

The American Mathematical Monthly, February 1984, Volume 91, Number 2, p. 139.

In most calculus books there is little effort given to showing that the cylindrical shell method and disk method give the same value when computing the volume of a solid of revolution. Indeed it is not obvious that these two distinct methods should give the same result. In some texts this is demonstrated when the trapezoid bounded by the x -axis, $y = mx + b$, $x = a$ and $x = b$ is revolved about the y -axis.

In this paper we shall show that the cylindrical shell and disk methods give the same value if the region revolved about the y -axis is bounded by $y = f(x)$, $x = a$, $x = b$ and the x -axis, provided $f(x)$ is a differentiable function on $[a, b]$ and $f(x)$ is one-to-one. The proof is simple and uses two theorems which the students have recently learned (substitution formula and integration by parts.) This proof can easily be included in a calculus course.

Consider the solid of revolution K produced by revolving the region bounded by $y = f(x)$, $x = a$, $x = b$ and the x -axis, about the y -axis. We use the shell method, which involves summing the volumes of cylindrical shells, to define the volume of K to be $\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n 2\pi x_i f(x_i) \Delta x_i$. If $f(x)$ is differentiable on $[a, b]$ and hence continuous there, this limit exists and is equal to $\int_a^b 2\pi x f(x) dx$.

Suppose the region is bounded by the function $x = g(y)$, $y = c$, $y = d$ and the y -axis. In the disk method, which involves summing the volumes of disks, we consider

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n \pi [g(y_i)]^2 \Delta y_i.$$

If $g(y)$ is continuous on $[c, d]$, this limit exists and is equal to $\int_c^d \pi [g(y)]^2 dy$.

THEOREM. *Let $0 \leq a < b$, and let $y = f(x)$ be differentiable, nonnegative and $1 - 1$ on $[a, b]$, with $f(a) = c$ and $f(b) = d$ where $c < d$.* Also let $f'(x)$ be continuous on $[a, b]$ and let $x = g(y)$ iff $y = f(x)$. If R is the region bounded by $y = f(x)$, the x -axis, $x = a$ and $x = b$ and R is revolved about the y -axis, then the value obtained by using the disk method is equal to the value obtained by using the cylindrical shell method. Equivalently $\pi[b^2d - a^2c] - \int_c^d \pi [g(y)]^2 dy = \int_a^b 2\pi x f(x) dx$.*

Proof: The region R can also be described as the region bounded by $x = m(y)$, $x = b$, $y = 0$ and $y = d$, where

$$m(y) = \begin{cases} a & \text{if } 0 \leq y < c, \\ g(y) & \text{if } c \leq y \leq d. \end{cases}$$

We observe from the way that $m(y)$ is defined that it is continuous on $[0, d]$. If we evaluate the volume obtained by revolving the region R about the y -axis by using the disk method, we find this to be $\int_0^d \pi b^2 dy - \int_0^c \pi a^2 dy - \int_c^d \pi [g(y)]^2 dy$. This is equal to

$$\int_0^d \pi b^2 dy - \int_0^c \pi a^2 dy - \int_c^d \pi [g(y)]^2 dy = \pi[b^2d - a^2c] - \int_c^d \pi [g(y)]^2 dy.$$

By the substitution formula, the latter expression is equal to

$$\pi[b^2d - a^2c] - \int_a^b \pi x^2 f'(x) dx.$$

A straightforward application of the integration by parts formula and algebraic simplification shows that

$$\int_a^b 2\pi x f(x) dx = \pi[b^2d - a^2c] - \int \pi x^2 f'(x) dx,$$

and the argument is complete.

*The result is true in case $d < c$, but a slight alteration is needed in the argument.