More on Tabular Integration by Parts

Leonard Gillman, The High Road, Austin, TX 78746

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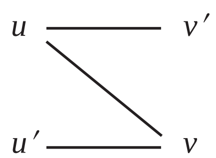
This note comments on the engaging article by David Horowitz on “Tabular Integration by Parts” [*College Mathematics Journal* 21 (1990) 307–311]. The method is based on iterating the diagram



The integral of this product
equals this product
minus the integral of this product,

and can be ended at any stage. Table 1 shows how the procedure handles the integral $\int x^2 \sin x\, dx$. A preliminary column lists alternating plus and minus signs, starting with plus. The integrand is written as a product of two factors, which head columns 1 and 2. Column 1 lists successive *derivatives* of the head entry, and column 2 successive *antiderivatives* of *its* head entry. The value of the integral is the sum of the indicated diagonal products, plus the integral of the product along the last row, all taken with the indicated signs. Note that if you reach a 0 in Column 1, as in the illustration, then the integral across that row provides the constant of integration.

Table 1

$$\int x^2 \sin x\, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

	Column 1	Column 2
$+$	x^2	$\sin x$
$-$	$2x$	$-\cos x$
$+$	2	$-\sin x$
$-$	0	$\cos x$

As Horowitz shows, the method is a blockbuster, as can be seen from the way it knocks off an integral like

$$\int \frac{12t^2 + 36}{\sqrt[5]{3t+2}}\, dt,$$

not to mention Taylor’s formula and the other examples he presents.

The purpose of this note is to call attention to a variant scheme, suggested in [Leonard Gillman and Robert H. McDowell, *Calculus*, W. W. Norton; 1st edition (1973) 328; 2nd edition (1978) 409], where it was used to integrate $\sec^3 x$ (but was not developed further). In the second edition this occurred *before* the section on integration by parts;

the spirit was "mess around and see what happens:" What do we differentiate in order to get $\sec^3 x$? Clearly it will arise when we differentiate $\sec x \tan x$; but so will another term that we don't want. Well, let's differentiate $\sec x \tan x$ anyway and then see if we can get rid of the unwanted term.

Table 2 shows the details of this scheme for $\int x^2 \sin x\, dx$, the problem treated above. The sum in the 2nd column is $x^2 \sin x$, the integrand, and the sum in the first column is its integral, so the answer is easy to read off. (One could build in a constant of integration by putting C in the first column and 0 opposite it in the second.) The scheme is really self-explanatory, but here are some comments. We write the integrand as

$$uv' \text{ (here } u = x^2,\ v' = \sin x),$$

then enter uv in column 1 and its derivative in column 2; this guarantees that the first term in column 2 will be the original integrand. Our task is then to cancel out the second term; we take it as our new uv' and continue the procedure.

Table 2

$\int x^2 \sin x\, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$	
uv	$uv' + u'v$
$-x^2 \cos x$	$x^2 \sin x - 2x \cos x$
$+2x \sin x$	$+ 2x \cos x + 2 \sin x$
$+2 \cos x$	$-2 \sin x$

This more contemplative scheme seems more informative than the other; you can see the mechanism, the work is very easy to check, and the final answer is very easy to read off. A disadvantage is that there is more writing to do, in fact about twice as much.

A mild point shows up in evaluating integrals such as $\int e^x \sin x\, dx$, where ordinarily one integrates by parts until the original integrand reappears on the other side of the equation, then transposes and solves. In the variant scheme the integrand comes back on the same side of the equation, so that what is taking place remains somewhat more transparent; see Table 3. (In the second step, we write the $e^x \cos x$ first because it is the uv' term.)

Table 3

$\int e^x \sin x\, dx = \frac{1}{2} e^x(\sin x - \cos x) + C$	
uv	$uv' + u'v$
$-e^x \cos x$	$e^x \sin x - e^x \cos x$
$+e^x \sin x$	$+ e^x \cos x$
	$+ e^x \sin x$

In the original method one still has to keep alert. A case in point is an integrand of the form $f(\ln x)$. Suppose we start off with

$+$	$f(\ln x)$	1
$-$	$\dfrac{f'(\ln x)}{x}$	x

Continuing the algorithm at this stage is unlikely to lead to anything except algebraic complication. Instead, we leave the product $xf(\ln x)$ untouched but rewrite the second line in the equivalent form

$$- \quad f'(\ln x) \quad 1$$

and continue from there. Table 4 shows the work for $\int (\ln x)^2\, dx$. The two lines marked * are identified, as are the two marked **. In this example, the variant scheme seems to be smoother (Table 5).

Table 4

$$\int (\ln x)^2\, dx = x[(\ln x)^2 - 2\ln x + 2] + C$$

	Column 1	Column 2	
$+$	$(\ln x)^2$	1	
$-$	$\dfrac{2\ln x}{x}$	x	*
$-$	$2\ln x$	1	*
$+$	$\dfrac{2}{x}$	x	**
$+$	2	1	**
$-$	0	x	

Table 5

$$\int (\ln x)^2\, dx = x[(\ln x)^2 - 2\ln x + 2] + C$$

uv	$uv' + u'v$
$(\ln x)^2(x)$	$(\ln x)^2 + 2\ln x$
$-(2\ln x)(x)$	$-2\ln x - 2$
$+\ 2x$	$+\ 2$

Another textbook favorite is

$$\int \sin \ln x \, dx.$$

The original method runs into the same problem as the preceding example; but the variant, Table 6, is smooth and in fact is isomorphic to Table 3.

Table 6

$\int \sin \ln x \, dx = \frac{1}{2} x(\sin \ln x - \cos \ln x) + C$	
uv	$uv' + u'v$
$(\sin \ln x)(x)$	$\sin \ln x + \cos \ln x$
$-(\cos \ln x)(x)$	$-\cos \ln x$
	$+\sin \ln x$