Integrals of Products of Sine and Cosine with Different Arguments

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Nearly every calculus text I have encountered in the past several years uses the identities

$$\cos(ax)\cos(bx) = \tfrac{1}{2}[\cos((a+b)x) + \cos((a-b)x)],$$
$$\sin(ax)\cos(bx) = \tfrac{1}{2}[\sin((a+b)x) + \sin((a-b)x)],$$
$$\sin(ax)\sin(bx) = \tfrac{1}{2}[\cos((a-b)x) - \cos((a+b)x)]$$

to evaluate integrals of the form

$$\int \cos(ax)\cos(bx)\,dx,$$

$$\int \sin(ax)\cos(bx)\,dx,$$

$$\int \sin(ax)\sin(bx)\,dx.$$

Most students balk in anticipation of more formulas to memorize.

These integrals are typically found in the section of a text dealing with integrating powers of trigonometric functions, which follows the section on integration by parts. I contend that these integrals should be done by repeated (iterated) integration by parts, just as integrals of the form $\int e^{kx}\cos(ax)\,dx$. Although not so easy as using the above identities, integration by parts is not difficult. For example consider the integral

$$I = \int \sin(2x)\cos(3x)\,dx.$$

Let $u = \sin(2x)$ and $dv = \cos(3x)\,dx$. Then $du = 2\cos(2x)\,dx$, and $v = \frac{1}{3}\sin(3x)$. Thus

$$I = \frac{1}{3}\sin(2x)\sin(3x) - \frac{2}{3}\int \cos(2x)\sin(3x)\,dx.$$

Now let $p = \cos(2x)$ and $dq = \sin(3x)\,dx$. Then $dp = -2\sin(2x)\,dx$, and $q = -\frac{1}{3}\cos(3x)$ yielding

$$I = \frac{1}{3}\sin(2x)\sin(3x) - \frac{2}{3}\left(-\frac{1}{3}\cos(2x)\cos(3x) - \frac{2}{3}I\right),$$

$$I = \frac{1}{3}\sin(2x)\sin(3x) + \frac{2}{9}\cos(2x)\cos(3x) + \frac{4}{9}I,$$

$$\frac{5}{9}I = \frac{1}{3}\sin(2x)\sin(3x) + \frac{2}{9}\cos(2x)\cos(3x) + C.$$

Finally,

$$\int \sin(2x)\cos(3x)\,dx = \frac{3}{5}\sin(2x)\sin(3x) + \frac{2}{5}\cos(2x)\cos(3x) + C.$$

For the student who has been taught tabular integration by parts the calculation runs as follows:

u	dv	
$\sin(2x)$	$\cos(3x)$	$I = \frac{1}{3}\sin(2x)\sin(3x) + \frac{2}{9}\cos(2x)\cos(3x) + \frac{4}{9}I,$
$2\cos(2x)$	$\frac{1}{3}\sin(3x)$	$\frac{5}{9}I = \frac{1}{3}\sin(2x)\sin(3x) + \frac{2}{9}\cos(2x)\cos(3x) + C$
$-4\sin(2x)$	$-\frac{1}{9}\cos(3x)$	$I = \frac{3}{5}\sin(2x)\sin(3x) + \frac{2}{5}\cos(2x)\cos(3x) + C.$

The integral is evaluated without the use of trigonometric identities and, as I prefer, in terms of the arguments of the trigonometric functions found in the original problem. As Grant [Moments on a rose petal, *CMJ* (1990) 225-227] mentions, when the result is in terms of the original arguments, checking an integral by differentiation is a viable option, even for the more complex integrals $\int \sin\theta \sin^n(m\theta)d\theta$ and $\int \cos\theta \sin^n(m\theta)d\theta$ which Grant tackles using integration by parts. (Incidentally, checking the example above and a few others by differentiation may prompt some to notice the forms that appear as antiderivatives and thereby to sense the possibility of yet another method: undetermined coefficients.)