A Geometric Proof of $\lim_{d \to 0^+} (-d \ln d) = 0$

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Almost every calculus book contains the example

$$\lim_{d \to 0^+} (-d \ln d) = 0,$$

and the computation of this limit is usually done by applying L'Hôpital's Rule. However, it can be viewed geometrically as follows: the portion of the unit square lying to the right of the vertical line $x = d$ and under the hyperbola $xy = d$ has area $-d \ln d$, (see Figure 1). As $d \to 0$ the area of this region will shrink to zero (see animation in Figures 2–5). The area of the region in Figure 1 is:

$$\int_d^1 \frac{d}{x}\, dx = d \ln x \Big|_{x=d}^{x=1} = -d \ln d.$$

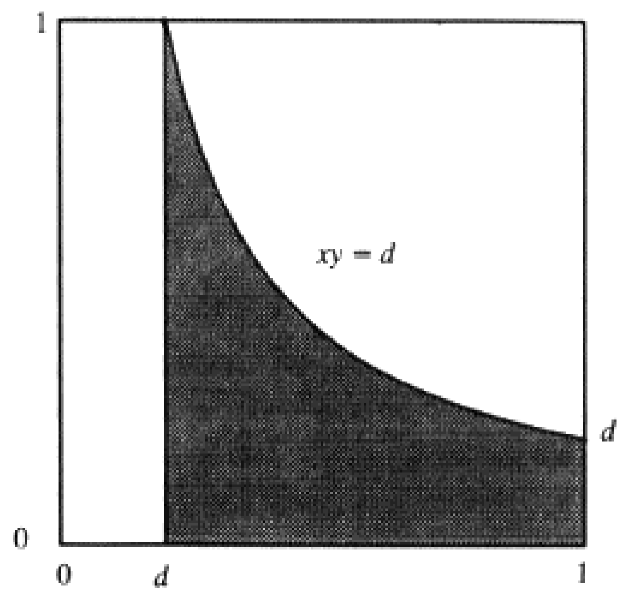

Figure 1
The region whose area is $-d \ln d$.

The sequence of graphical animations corresponding to $d = 0.1, 0.05, 0.01$ and 0.0005 are shown in Figures 2–5.

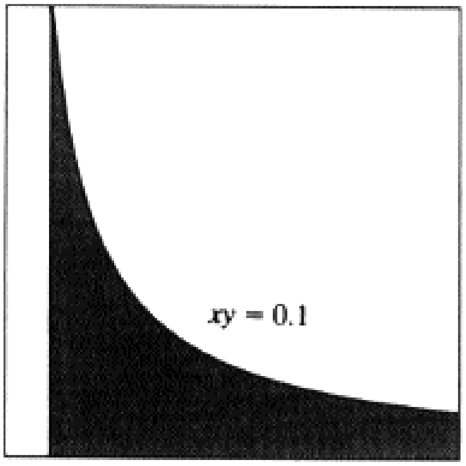

Figure 2
area $= -0.1\ln(0.1)$.

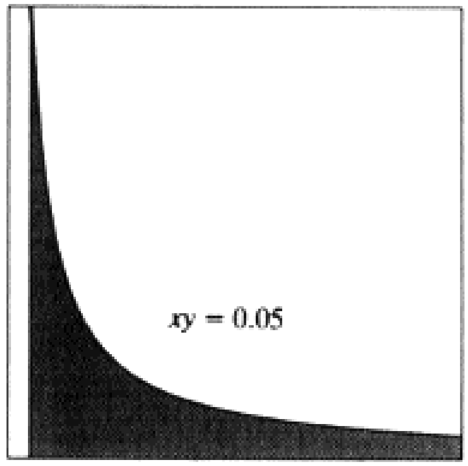

Figure 3
area $= -0.05\ln(0.05)$.

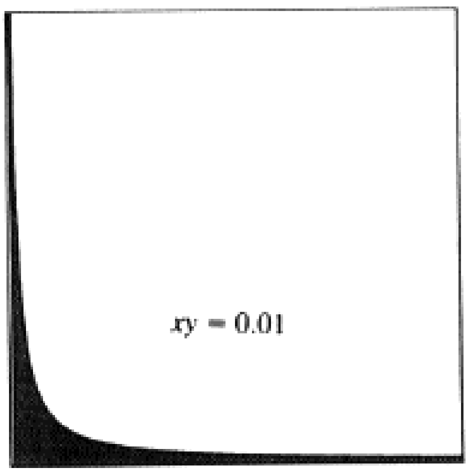

Figure 4
area $= -0.01\ln(0.01)$.

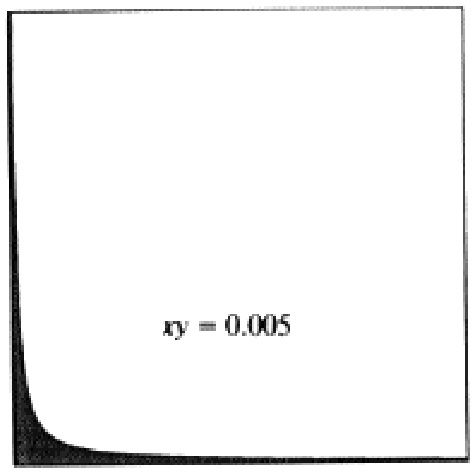

Figure 5
area $= -0.005\ln(0.005)$.