Six Ways to Sum a Series

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Dan Kalman This fall I have joined the mathematics faculty at American University, Washington D. C. Prior to that I spent 8 years at the Aerospace Corporation in Los Angeles, where I worked on simulations of space systems and kept in touch with mathematics through the programs and publications of the MAA. At a national meeting I heard the presentation by Zagier referred to in the article. Convinced that this ingenious proof should be more widely known, I presented it at a meeting of the Southern California MAA section. Some enthusiastic members of the audience then shared their favorite proofs and references with me. These led to more articles and proofs, and brought me into contact with a realm of mathematics I never guessed existed. This paper is the result.

The concept of an infinite sum is mysterious and intriguing. How can you add up an infinite number of terms? Yet, in some contexts, we are led to the contemplation of an infinite sum quite naturally. For example, consider the calculation of a decimal expansion for $1/3$. The long division algorithm generates an endlessly repeating sequence of steps, each of which adds one more 3 to the decimal expansion. We imagine the answer therefore to be an endless string of 3's, which we write 0.333. . .. In essence we are defining the decimal expansion of $1/3$ as an infinite sum

$$1/3 = 0.3 + 0.03 + 0.003 + 0.0003 + \cdots.$$

For another example, in a modification of Zeno's paradox, imagine partitioning a square of side 1 as follows: first draw a diagonal line that cuts the square into two triangular halves, then cut one of the halves in half, then cut one of these halves in half, and so on ad infinitum. (See Figure 1.) Then the area of the square is the sum of the areas of all the pieces, leading to another infinite sum

$$1 = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots.$$

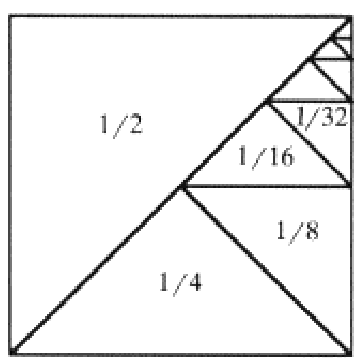

Figure 1
Partitioned unit square.

Although these examples illustrate how naturally we are led to the concept of an infinite sum, the subject immediately presents difficult problems. It is easy to describe an infinite series of terms, much more difficult to determine the sum of the series. In this paper I will discuss a single infinite sum, namely, the sum of the squares of the reciprocals of the positive integers. In 1734 Leonhard Euler was the first to determine an exact value for this sum, which had been considered actively for at least 40 years. By today's standards, Euler's proof would be considered unacceptable, but there is no doubt that his result is correct. Logically correct proofs are now known, and indeed, there are many different proofs that use methods from seemingly unrelated areas of mathematics. It is my purpose here to review several of these proofs and a little bit of the mathematics and history associated with the sum.

Background

It is clear that when an infinite number of positive quantities are added, the result will be infinitely large unless the quantities diminish in size to zero. One of the simplest infinite sums that has this property is the *harmonic* series,

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \cdots.$$

It may come as a surprise that this sum becomes infinitely large (that is, it *diverges*). To see this, we ignore the first term of the sum and group the remaining terms in a special way: the first group has 1 term, the next group has 2 terms, the next group 4 terms, and next 8 terms, and so on. The first several groups are depicted below:

$$\begin{gathered} \tfrac{1}{2} \\ \tfrac{1}{3} + \tfrac{1}{4} > \tfrac{1}{4} + \tfrac{1}{4} \\ \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} > \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} \\ \tfrac{1}{9} + \tfrac{1}{10} + \tfrac{1}{11} + \tfrac{1}{12} + \tfrac{1}{13} + \tfrac{1}{14} + \tfrac{1}{15} + \tfrac{1}{16} > \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16}. \end{gathered}$$

The inequalities are derived by observing that in each group the last term is the smallest, so that repeatedly adding the last term results in a smaller sum than adding the actual terms in the group. Now notice that in each case, the right-hand side of the inequality is equal to $1/2$. Thus, when the terms are grouped in this way, we see that the sum is larger than adding an infinite number of $1/2$'s, which is, of course, infinite.

We may conclude that although the terms of he harmonic series dwindle away to 0, they don't do it fast enough to produce a finite sum. On the other hand, we have already seen that adding all the powers of $1/2$ does produce a finite sum. That is,

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots = 1.$$

(More generally, for any $|z| < 1$, the geometric series $1 + z + z^2 + z^3 + \cdots$ adds up $1/(1 - z)$). Apparently, these terms get small so fast that adding an infinite number of them still produces a finite result. It is natural to wonder what happens for a sum that falls between these two examples, with terms that decrease more rapidly than the harmonic series, but not so rapidly as the geometric series. An obvious example comes readily to hand, the sum of the *squares* of the reciprocals of the integers: $1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \cdots$. For reference, we will call this Euler's series. Does the sum get infinitely large? The answer is no, which can be seen as follows. We are interested in the sum

$$1+\frac{1}{2\cdot 2}+\frac{1}{3\cdot 3}+\frac{1}{4\cdot 4}+\frac{1}{5\cdot 5}+\cdots.$$

This is evidently less than the sum

$$1+\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots.$$

Now rewrite each fraction as a difference of two fractions. That is,

$$\frac{1}{1\cdot 2}=\frac{1}{1}-\frac{1}{2}$$

$$\frac{1}{2\cdot 3}=\frac{1}{2}-\frac{1}{3}$$

$$\frac{1}{3\cdot 4}=\frac{1}{3}-\frac{1}{4}$$

$$\frac{1}{4\cdot 5}=\frac{1}{4}-\frac{1}{5}$$

$$\vdots$$

Substitute these values into the sum and we obtain

$$1+\tfrac{1}{1}-\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{3}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{4}-\tfrac{1}{5}+\cdots.$$

If we add all the terms of this last sum, the result is 2. So we may conclude at least that the sum we started with, $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots$, is less than 2. This implies that the terms actually add up to some definite number. But which one?

Before proceeding, let us take another look at the two arguments advanced above, the first for the divergence of the harmonic series, and the second for the convergence of Euler's series. It mighty appear at first glance that we have indulged in some mathematical sleight of hand. The two arguments are of such different flavors. It seems unfair to apply different methods to the two series and arrive at different conclusions, as if the conclusion is a consequence of the method rather than an inherent property of the series. If we applied the divergence argument to Euler's series, might we then arrive at the conclusion that it diverges? This is an instructive exercise, and the reader is encouraged to undertake it.

We return to the question, what is the sum of Euler's series? Of course, you can use a calculator to estimate the sum. Adding up 10 terms gives 1.55, but that doesn't tell us much. The correct two decimal approximation is 1.64, and is not reached until after more than 200 terms. And even then it is not at all obvious that the first two decimal places are correct. Compare the case of the harmonic series which we know has an infinite sum. After 200 terms of that series, the total is still less than 6. For these reasons, direct calculation is not very helpful.

It is possible to make accurate estimates of the sum by using methods other than direct calculation. On a very elementary level, by comparing a single term $1/n^2$ with $\int_n^{n+1} dx/x^2$, the methods of calculus can be used to show that

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} + \frac{1}{n+1}$$

is a much better approximation to the full total than just using the first n or $n + 1$ terms. In fact, with this approximation, the error must be less than $1/n(n+1)$. Taking $n = 14$, for example, the approximation will be accurate to two decimal places. This is a big improvement on adding up 200 terms, and not knowing even then if the first two decimals are correct.

Calculating the first few decimal places of the sum of Euler's series was a problem of some interest in Euler's time. He himself worked on the problem, obtaining approximation formulas that allowed him to determine the first several decimal places, in the same way that the approximation and error estimate were used in the preceding paragraph. Later, Euler derived an exact value for the sum. Erdös and Dudley [**5**] describe Euler's contribution this way:

> In 1731 he obtained the sum accurate to 6 decimal places, in 1733 to 20, and in 1734 to infinitely many. . . .

A more detailed history of this problem, and of Euler's contribution are presented in [**4**]. Briefly, Oresme showed the divergence of the harmonic series in the 14th century. In 1650, Mengali asked whether Euler's series converges. In 1655 John Wallis worked on the problem, as did John Bernoulli in 1691. Thus, when Euler published his value for the sum in 1734, the problem had already been worked on by formidable mathematicians for several decades. By an ingenious application of formal algebraic methods, Euler derived the value of the sum to be $\pi^2/6$.

Euler's Proof

As mentioned earlier, Euler's proof is not considered valid today. Nevertheless, it is quite interesting, and worth reviewing here. Actually, Euler gave several proofs over a number of years, including two in the paper of 1734 [**6**]. What we present here is essentially the same as the argument given in sections 16 and 17 of that paper, and is in the same form as in [**8**] and [**18**]. The basic idea is to obtain a power series expansion for a function whose roots are multiples of the perfect squares 1, 4, 9, etc. Then we apply a property of polynomials to obtain the sum of the reciprocals of the roots. The other derivation given in Euler's 1734 paper is discussed in [**4**, section 4] and [**10**, pp. 308–309].

Here is the argument: The sine function can be represented as a power series

$$\sin x = x - \frac{x^3}{3 \cdot 2} + \frac{x^5}{5 \cdot 4 \cdot 3 \cdot 2} - \frac{x^7}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} + \cdots$$

which we think of as an infinite polynomial. Divide both sides of this equation by x and we obtain an infinite polynomial with only even powers of x; replace x with $\sqrt{x}$ and the result is

$$\frac{\sin\sqrt{x}}{\sqrt{x}} = 1 - \frac{x}{3 \cdot 2} + \frac{x^2}{5 \cdot 4 \cdot 3 \cdot 2} - \frac{x^3}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} + \cdots.$$

We will call this function f. The roots of f are the numbers $\pi^2, 4\pi^2, 9\pi^2, 16\pi^2, \ldots$. Note that 0 is not a root, because there the left-hand side is undefined, while the right-hand side is clearly 1.

Now Euler knew that adding up the reciprocals of all the roots of a polynomial results in the negative of the ratio of the linear coefficient to the constant coefficient. In symbols, if

$$(x - r_1)(x - r_2)\ldots(x - r_n) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \tag{1}$$

then

$$\frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_n} = -a_1/a_0.$$

Assuming that the same law must hold for a power series expansion, he applied it to the function f, concluding that

$$\frac{1}{6} = \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \cdots.$$

Multiplying both sides of this equation by π^2 yields $\pi^2/6$ as the sum of Euler's series.

Why is this not considered a valid proof today? The problem is that power series are *not* polynomials, and do not share all the properties of polynomials. To get an understanding of the property that Euler used, that the reciprocals of a polynomial's roots add up to the negative ratio of the two lowest order coefficients, let us consider a polynomial of degree 4. Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

have roots r_1, r_2, r_3, r_4. Then

$$p(x) = (x - r_1)(x - r_2)(x - r_3)(x - r_4).$$

If we multiply out the factors at the right, we find that

$$a_0 = r_1r_2r_3r_4$$
$$a_1 = -r_2r_3r_4 - r_1r_3r_4 - r_1r_2r_4 - r_1r_2r_3.$$

From these it is clear that

$$-a_1/a_0 = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}.$$

A similar argument works for a polynomial of any degree.

Notice that this argument would not work for an infinite polynomial without, at the very least, some theory of infinite products. In any case, the result does not apply to all power series. For example, the identity

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

holds for all x of absolute value less than 1. Now consider the function $g(x) = 2 - 1/(1 - x)$. Clearly, g has a single root, $1/2$. The power series expansion for $g(x)$ is $1 - x - x^2 - x^3 - \cdots$, so $a_0 = 1$ and $a_1 = -1$. The sum of the reciprocal roots does not equal the ratio $-a_1/a_0$. While this example shows that the reciprocal root sum law cannot be applied blindly to all power series, it does not imply that the law never holds. Indeed, the law must hold for the function $f(x) = \sin\sqrt{x}/\sqrt{x}$ because we have independent proofs of Euler's result. Notice the differences between this f and the g of the counterexample. The function f has an infinite number of roots, where g has but one. And f has a power series that converges for all x, where the series g converges only for $-1 < x < 1$. Is there a theorem that provides conditions under which a power series satisfies the reciprocal root sum law? I don't know.

Euler's proof is generally conceded not to hold up to today's standards. There are a number of proofs that are considered acceptable, and they display a wide variety of methods and approaches. Shortly we will cover several of these proofs. However, before leaving Euler, two more points deserve mention. First, the aspect of Euler's methods that are considered invalid today generally involve the informal and intuitive way he manipulated the infinitely large and small. The modern subject of nonstandard analysis has provided in our time what Euler lacked in his: a sound treatment of analysis using infinite and infinitesimal quantities. The methods of nonstandard analysis have been used to validate some of Euler's arguments. That is, it has been possible to develop logically correct arguments that are conceptually the same as Euler's. In [**12**], for example, Euler's derivation of an infinite product for the sine function is made rigorous. This product formula is closely related to Euler's argument traced above. Euler gave another proof in 1748, again by comparing a power series to an infinite product. This argument has also been made rigorous using nonstandard analysis [**14**].

The second point I wish to make is that Euler was able to generalize his methods to many other sums. In particular, he developed a formula that gives the sum $1 + 1/2^s + 1/3^s + 1/4^s + \cdots$ for any even power s. The idea of allowing the power s to vary prompts the definition of a function of s: $\zeta(s) = 1 + 1/2^s + 1/3^s + 1/4^s + \cdots$. This is called the Riemann zeta function, and it has great significance in number theory. When s is an even integer, Euler's formula gives the value of $\zeta(s)$ as a rational nultiple of π^s. Interestingly, while the zeta function values are known exactly for the even integers, things are much more obscure for the odd integers. For example, it was not even known for sure that $\zeta(3)$ is irrational until 1978. An interesting account of this discovery can be found in [**19**]. The November 1983 issue of *Mathematics Magazine* is devoted to articles on Euler, [**10**] being one example.

Modern Proofs

Let us turn now to the modern proofs of Euler's result. We will consider five different approaches. The first proof uses no mathematics more advanced than trigonometry. It is not as spectacular as some of the other proofs, in that it doesn't really have strange twists or connections to other areas of mathematics. On the other hand, it generalizes in a direct way to derive Euler's formula for $\zeta(2n)$. The second proof is based on methods of calculus, and involves a sequence of transformations that will take your breath away. Next, we will enter the realm of complex analysis and use a method called contour integration. The fourth proof, also in the complex world, involves techniques from Fourier analysis. Finally, we finish with a proof based on formal manipulations that Euler himself would have been proud of. This last approach uses both complex numbers and elementary calculus. In the middle of this sequence of proofs we will take a brief time out for an application.

Complex numbers show up repeatedly in these proofs, so it is appropriate here to remember a few elementary properties. Most important is the identity $e^{ix} = \cos x + i \sin x$, along with the special cases $e^{i\pi} = -1$ and $e^{in\pi} = (-1)^n$. Raising both sides of the general identity to the n^{th} power produces de Moivre's theorem: $\cos nx + i \sin nx = (\cos x + i \sin x)^n$. By expanding the power on the right and then gathering real and complex parts, formulas for $\cos nx$ and $\sin nx$ are obtained. For a complex number $x + iy$, the absolute value is defined as $|x + iy| = \sqrt{x^2 + y^2}$ and the conjugate is $\overline{x + iy} = x - iy$. If (r, θ) are the polar coordinates for (x, y), then $x + iy = re^{i\theta}$.

It will also be necessary to use the familiar sigma notation

$$\sum_{k=1}^{\infty} f(k) = f(1) + f(2) + f(3) + \cdots,$$

which renders Euler's result as

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Trigonometry and Algebra. The first proof, published by Papadimitriou [**15**], depends on a special trigonometric identity. Once the identity is known, the derivation of Euler's result is fairly direct and unsurprising. Apostol [**2**] generalizes this proof to compute the formula for $\zeta(2n)$. A closely related proof is given by Giesy [**7**]. Note that Apostol and Giesy each give several additional references to elementary derivations of Euler's result.

The trigonometric identity involves the angle $\omega = \pi/(2m + 1)$, and several of its multiples. The identity reads

$$\cot^2 \omega + \cot^2(2\omega) + \cot^2(3\omega) + \cdots + \cot^2(m\omega) = \frac{m(2m - 1)}{3}. \tag{2}$$

For example, with $m = 3$ we have $\omega = \pi/7$ and the identity reads

$$\cot^2 \omega + \cot^2(2\omega) + \cot^2(3\omega) = 5.$$

We will use identity (2) to derive the sum of Euler's series, and then discuss the derivation of the identity.

For any x between 0 and $\pi/2$, the following inequality holds.

$$\sin x < x < \tan x$$

Squaring and inverting each term in the inequality leads to

$$\cot^2 x < \frac{1}{x^2} < 1 + \cot^2 x.$$

Now to use (2), we will successively replace x in this inequality by ω, 2ω, 3ω, and so on, and sum the results. This gives

$$\begin{aligned} &\cot^2 \omega + \cot^2(2\omega) + \cot^2(3\omega) + \cdots + \cot^2(m\omega) \\ &< 1/\omega^2 + 1/4\omega^2 + 1/9\omega^2 + \cdots + 1/m^2\omega^2 \\ &< m + \cot^2 \omega + \cot^2(2\omega) + \cot^2(3\omega) + \cdots + \cot^2(m\omega). \end{aligned}$$

Using identity (2) then produces

$$\frac{m(2m-1)}{3} < \frac{1}{\omega^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{m^2}\right) < \frac{m(2m-1)}{3} + m.$$

For a final transformation, multiply through by ω^2 and substitute $\omega = \pi/(2m+1)$:

$$\frac{m(2m-1)\pi^2}{3(2m+1)^2} < 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{m^2} < \frac{m(2m-1)\pi^2}{3(2m+1)^2} + \frac{m\pi^2}{(2m+1)^2}.$$

This final set of inequalities provides upper and lower bounds for the sum of the first m terms of Euler's series. Now let m go to infinity. The lower bound is

$$\frac{m(2m-1)\pi^2}{3(2m+1)^2} = (\pi^2/6)\,\frac{2m^2 - m}{2m^2 + 2m + 0.5}$$

which approaches $\pi^2/6$. At the same time, the upper bound also approaches $\pi^2/6$ as its second term decreases to 0. Euler's sum is squeezed in between these bounds, and so it must equal $\pi^2/6$ as well.

This completes the proof of Euler's result, subject to the validity of identity (2). For completeness, we will prove that next. Interestingly enough, the derivation uses a property of polynomials very similar to the one used in Euler's proof above. Specifically, for any polynomial

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

the sum of the roots is just $-a_{n-1}/a_n$. The derivation of this property is so similar to the previously given proof of the reciprocal root sum law that it is recommended as an exercise for the reader. We will use the property by considering a polynomial whose roots are the terms $\cot^2(k\omega)$ on the left side of (2). Equating the sum of the roots to the negative ratio of the two highest order coefficients will yield the desired identity.

The polynomial is generated by manipulating de Moivre's identity with n odd. Considering just the imaginary parts of each side of the identity, we begin with

$$\sin n\theta = \binom{n}{1}\sin\theta\cos^{n-1}\theta - \binom{n}{3}\sin^3\theta\cos^{n-3}\theta + \cdots \pm \sin^n\theta$$

$$= \sin^n\theta\left(\binom{n}{1}\cot^{n-1}\theta - \binom{n}{3}\cot^{n-3}\theta + \cdots \pm 1\right).$$

Assuming that $0 < \theta < \pi/2$, we may divide through by $\sin^n\theta$ to obtain

$$\frac{\sin n\theta}{\sin^n\theta} = \binom{n}{1}\cot^{n-1}\theta - \binom{n}{3}\cot^{n-3}\theta + \cdots \pm 1.$$

Now n is odd, so $n-1$ is even. Let us express the exponents on the right side of the preceding equation in terms of $m = (n-1)/2$:

$$\frac{\sin n\theta}{\sin^n\theta} = \binom{n}{1}\cot^{2m}\theta - \binom{n}{3}\cot^{2m-2}\theta + \cdots \pm 1.$$

This is where we see the polynomial emerge. Make the substitution $x = \cot^2\theta$ and we have

$$\frac{\sin n\theta}{\sin^n \theta} = \binom{n}{1}x^m - \binom{n}{3}x^{m-1} + \cdots \pm 1.$$

At the right is a polynomial; we can read off the two leading coefficients. The expression at the left reveals to us m distinct roots. Indeed $\sin n\theta = 0$ for $\theta = \pi/n$, $2\pi/n, \ldots, m\pi/n$, so we would like to conclude that $x = \cot^2\pi/n, \cot^2 2\pi/n, \ldots, \cot^2 m\pi/n$ are m distinct roots of the polynomial. It will suffice to verify that all of the θ's are strictly between 0 and $\pi/2$ since then they generate distinct positive values of the cotangent function. Remembering that $n = 2m + 1$, we see that the largest θ is $\pi \cdot m/(2m + 1)$ which is evidently less than $\pi/2$.

From this analysis, we conclude that the polynomial $\binom{n}{1}x^m - \binom{n}{3}x^{m-1} + \cdots \pm 1$ has the roots $\cot^2\pi/n, \cot^2 2\pi/n, \cot^2 3\pi/n, \ldots, \cot^2 m\pi/n$. The sum of these roots is the negative ratio of the two leading coefficients: $\binom{n}{3}/\binom{n}{1}$. To complete the derivation, we set $\omega = \pi/(2m + 1)$ and compute

$$\begin{aligned}
\cot^2\omega + \cot^2(2\omega) + \cdots + \cot^2(m\omega) &= \frac{\binom{n}{3}}{\binom{n}{1}} \\
&= \frac{n(n-1)(n-2)/6}{n} \\
&= \frac{(n-1)(n-2)}{6} \\
&= \frac{2m(2m-1)}{6} \\
&= \frac{m(2m-1)}{3}.
\end{aligned}$$

This completes the first proof. Although it is fairly direct, it requires the use of an obscure identity. Other than that, nothing more difficult than high school trigonometry is required, and there is nothing particularly surprising or exciting about the argument. The next proof provides a dramatic contrast. It uses methods of calculus, and makes several surprising and unexpected transformations.

Odd terms, Geometric Series, and a Double Integral. The next proof is one I originally saw presented in a lecture by Zagier [**20**]. He mentioned that the proof was shown to him by a colleague who had learned of it through the grapevine. It is closely related to a proof given by Apostol [**3**], but has a couple of unique twists. I have not seen this proof in print.

It will simplify the discussion to let E represent $\sum_{k=1}^{\infty} 1/k^2$. The point of the proof is then to show that $E = \pi^2/6$. We begin with just the even terms of the sum. Observe:

$$\begin{aligned}\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\cdots &= \sum_{k=1}^{\infty}\frac{1}{(2k)^2}\\ &= \sum_{k=1}^{\infty}\frac{1}{4k^2}\\ &= \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}\\ &= \frac{1}{4}E.\end{aligned}$$

Since the even terms add up to one fourth of the total, the odd terms must account for the remaining three fourths. Write this in equation form as

$$\frac{3}{4}E = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}. \tag{3}$$

Now we shift gears. Consider the following definite integral:

$$\begin{aligned}\int_0^1 x^{2k}dx &= \left.\frac{x^{2k+1}}{2k+1}\right|_0^1\\ &= \frac{1}{2k+1}.\end{aligned}$$

Of course, this equation would be just as correct if we used the variable y in place of x. Therefore we may write

$$\begin{aligned}\left(\frac{1}{2k+1}\right)^2 &= \int_0^1 x^{2k}\,dx\int_0^1 y^{2k}\,dy\\ &= \int_0^1\int_0^1 x^{2k}y^{2k}\,dx\,dy\end{aligned}$$

and this is substituted in equation (3) to obtain

$$\frac{3}{4}E = \sum_{k=0}^{\infty}\int_0^1\int_0^1 x^{2k}y^{2k}\,dx\,dy.$$

For the next step, exchange the sum and the double integral to obtain

$$\frac{3}{4}E = \int_0^1\int_0^1\sum_{k=0}^{\infty} x^{2k}y^{2k}\,dx\,dy.$$

Concentrating on the sum part, notice that its terms are the powers of x^2y^2. The geometric series formula mentioned in the first section gives the total as $1/(1-x^2y^2)$, leading to

$$\frac{3}{4}E = \int_0^1\int_0^1\frac{1}{1-x^2y^2}\,dx\,dy.$$

To complete the derivation, we need only evaluate this double integral. An ingenious change of variables makes this step trivial. The substitution is given by $x = \sin u/\cos v$ and $y = \sin v/\cos u$. Applying the methods of multivariate calculus, we can show that $dx\,dy/(1-x^2y^2) = du\,dv$, and that the region of integration in terms of u and v is the triangle in the first quadrant illustrated in Figure 2. Therefore, the double integral yields the area of the triangle, $\pi^2/8$, which implies that

$$\frac{3}{4}E = \frac{\pi^2}{8}.$$

Thus, $E = \pi^2/6$, as required.

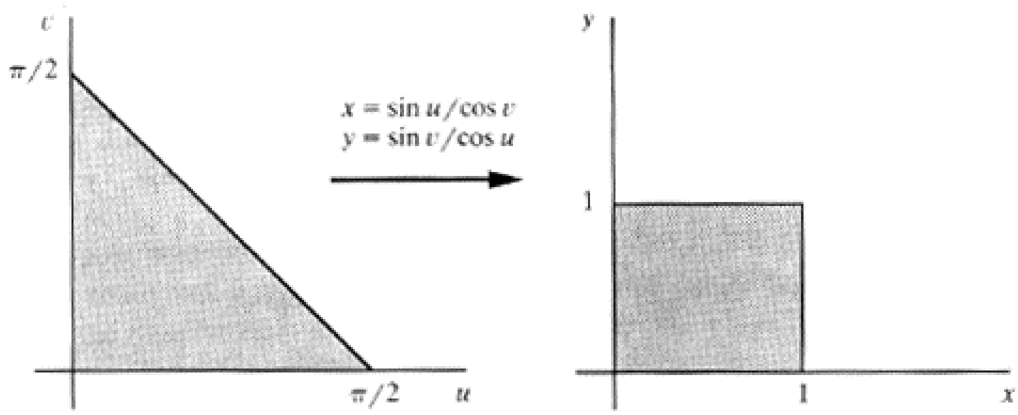

Figure 2
Transformed region of integration.

Two comments should be made here. First, interchanging the integral and the sum does require some justification. In Euler's day, the conditions under which such an operation is valid were not understood. Today the conditions are known and are generally considered in an advanced calculus course. In the case at hand, since $1/(1 - x^{2k}y^{2k})$ is positive at every point in the region of integration save $(1, 1)$, the monotone convergence theorem [**16**, Theorem 10.30] provides the necessary justification. One should also address the fact that the integrand in the original integral is undefined at one point of the region of integration; the usual methods for improper integrals apply.

Second, the change of variables in the double integral also requires a little work. Recall that the rule for transforming *dx dy* into an expression involving *du dv* depends on calculating the Jacobian of the transformation. And there is some effort involved in verifying that the change of variables transformation maps the triangle illustrated in *uv* space into the unit square in *xy* space.

Residue Calculus. The third proof applies a technique from complex analysis known as residue calculus. A full account of this technique can be found in any introductory text on complex analysis. For the present discussion the goal is simply an intuitive feel for the structure of the argument. For this purpose, we will discuss the basic ideas of residue calculus informally.

Residue calculus concerns functions with poles (which may be thought of as places where a denominator goes to 0) defined in the complex plane. Suppose that f is such a function, and has a pole at z_0. Then there is a power series expansion that describes how f behaves near z_0. It might look like this:

$$f(z_0 + z) = a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + \cdots.$$

The fact that there is a pole at z_0 is revealed by the negative powers of z. It is evident that as z goes to 0, $f(z_0 + z)$ blows up. In this example there are two terms with negative powers of z. In the general case, there may be any finite number of terms with negative powers of z.

A second central ingredient in residue calculus is the *complex integral*. For this discussion, the complex integral may be thought of as a kind of line integral. The integrand $f(z)\,dz$ is an exact differential if f is the derivative of a complex function throughout a region containing the path. The complex integral behaves like a line integral in that over a closed path, the integral of an exact differential is 0. In particular, we will consider a closed path that encloses 0, and for the integrand we take the expansion of $f(z_0 + z)$. Each term in the expansion is the derivative of a complex function, except for the term with exponent -1. This corresponds to the fact in real calculus that the antiderivative of x^k is $x^{k+1}/(k+1)$, except when $k = -1$. Of course, in the real case, we know that the antiderivative of $1/x$ is $\ln x$. Unfortunately, in the complex plane, it is not possible to define a natural logarithm consistently on any closed path encircling the origin. In fact, a line integral of z^{-1} around such a path does not produce 0, rather, it produces $2\pi i$. This actually makes good sense intuitively, if we think about how a complex natural logarithm should behave. In polar form, any complex number z can be expressed as $re^{i\theta} = e^{\ln r + i\theta}$ where (r, θ) are the usual polar coordinates for the point z in the complex plane. The natural logarithm should then be $\ln r + i\theta$. Now if we integrate $1/z$ along a path from z_1 to z_2, we expect the result to be $\ln r_2 + i\theta_2 - \ln r_1 - i\theta_1$. On our closed path, $z_1 = z_2$, and $r_2 - r_1 = 0$. But if we traverse the path once counterclockwise, varying θ continuously along the way, then $\theta_2 - \theta_1$ is 2π. Thus, the integral should produce a value of $2\pi i$. To generalize slightly, if we integrate $f(z_0 + z)$ along a path circling 0 once counterclockwise, every term of the sum vanishes except the z^{-1} term, and integrating that term results in $a_{-1} \cdot 2\pi i$. Since the contribution of the z^{-1} term is all that is left of f after integrating, the coefficient a_{-1} is called the residue of f at z_0.

The functions studied in the residue calculus might blow up at more than one place. For example, the function $1/(z^2 + 1)$ has poles at both i and $-i$. But if a function can always be expanded in a power series with a finite number of negative exponent terms, then the line integral about a simple closed path (in the counterclockwise direction) encircling a finite number of poles is equal to $2\pi i$ times the sum of the residues at those poles.

This is all very interesting, but what on earth does it have to do with Euler's sum? The answer is that using residue calculus, we can compute a sum by doing a complex integral. Actually, we will use a limiting argument involving a sequence of paths P_n. Each of these paths encloses a finite number of poles for our function $f(z)$, and the sum of the residues will include finitely many of the terms of Euler's sum. As n goes to infinity, two things will happen. First, the line integral of f over the path P_n will go to 0. But at the same time, the sum of the residues will approach an expression which contains all the terms of Euler's sum. Equating the sum of the residues to 0 then yields our final result.

The function used in this argument is $f(z) = \cot(\pi z)/z^2$. The path P_n is a rectangle centered at the origin with sides parallel to the real and imaginary axes in the complex plane (Figure 3). The sides intersect the real axis at $\pm(n + 1/2)$ and the imaginary

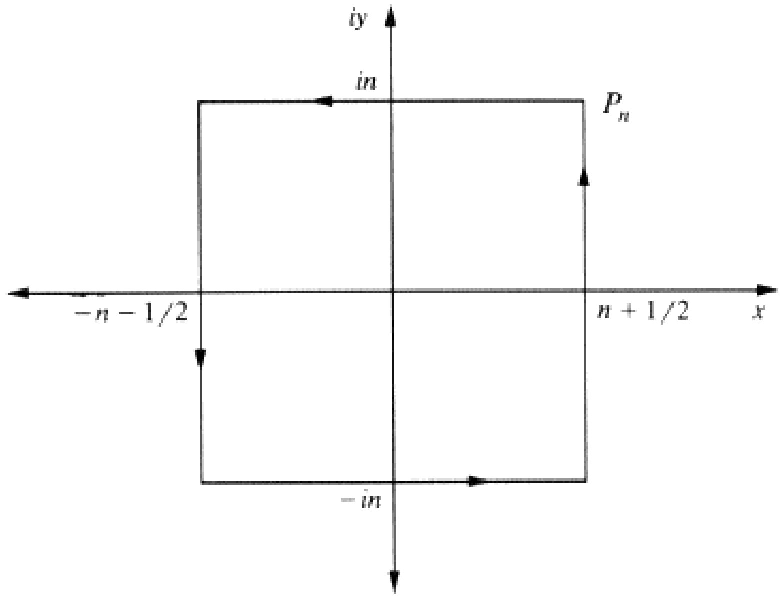

Figure 3
Path of integration.

axis at $\pm ni$. It may be shown that $|\cot(\pi z)| < 2$ for all z on the path P_n. (Actually, we can get a much more accurate bound than 2, but accuracy is not important here.) At the same time $|z| \geq n$ on the path, so $|f(z)| < 2/n^2$. Bounding $|f(z)|$ on the path in this way permits us to estimate the integral. We have

$$\left|\oint_{P_n} f(z)\, dz\right| < \frac{2}{n^2}(8n + 2)$$

where $8n + 2$ is the length of the path. Now it is clear that as n goes to infinity, the integral goes to 0.

To complete the argument, we observe that f has poles at each of the integers, and determine that the residue is $1/\pi k^2$ at $k \neq 0$, and $-\pi/3$ at 0. Before carrying through these calculations, let us see how the derivation of Euler's formula concludes. Since the integral over P_n goes to 0, we infer that $2\pi i$ times the sum of all the residues is 0. Combining the residues at k and $-k$ into a single term, this leads to

$$\frac{-2\pi^2 i}{3} + 4i\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right) = 0.$$

A trivial rearrangement of this equation reveals $E = \pi^2/6$.

All that remains of this proof is the calculation of the residues. For the residue at 0, let us observe that

$$z \cot(\pi z) = z\frac{\cos(\pi z)}{\sin(\pi z)}$$
$$= z\frac{1 - \pi^2 z^2/2 + \pi^4 z^4/24 - \cdots}{\pi z - \pi^3 z^3/6 + \pi^5 z^5/120 - \cdots}$$

$$= \frac{1 - \pi^2 z^2/2 + \pi^4 z^4/24 - \cdots}{\pi - \pi^3 z^2/6 + \pi^5 z^4/120 - \cdots}.$$

Using the long division algorithm, the ratio can be expressed as a power series. The first few terms are shown below.

$$z \cot(\pi z) = \frac{1}{\pi} - \frac{\pi z^2}{3} - \frac{\pi^3 z^4}{45} + \cdots$$

By dividing both sides of this equation by z^3, we derive

$$\frac{\cot(\pi z)}{z^2} = \frac{z^{-3}}{\pi} - \frac{\pi z^{-1}}{3} - \frac{\pi^3 z}{45} + \cdots.$$

Reading off the coefficient of z^{-1}, we see that the residue at 0 is $-\pi/3$.

We use a slightly different method for the residue at k. Suppose that we calculated the power series for $f(k + z)$ as

$$f(k + z) = a_{-1} z^{-1} + a_0 + a_1 z + a_2 z^2 + \cdots.$$

Then

$$zf(k + z) = a_{-1} + a_0 z + a_1 z^2 + a_2 z^3 + \cdots$$

and it is clear that

$$\begin{aligned} a_{-1} &= \lim_{z \to 0} [zf(k + z)] \\ &= \lim_{z \to 0} z \frac{\cot(\pi(k + z))}{(k + z)^2} \\ &= \lim_{z \to 0} \frac{z}{\sin(\pi(k + z))} \frac{\cos(\pi(k + z))}{(k + z)^2}. \end{aligned}$$

Apply L'Hôpital's rule to the first factor, and find $a_{-1} = 1/\pi k^2$. This gives the residue at k as previously asserted.

This calculation appears to rely on knowing in advance that the power series for $f(k + z)$ has only one term with a negative power of z. Why were there no terms involving z^{-2}, z^{-3}, as there were for the residue at 0? Actually, the answer is implicit in the limit we calculated above. Since $zf(k + z)$ has a limit at 0, its power series cannot have any terms with negative powers of z. Thus, every term of the series for $f(k + z)$ must have an exponent of at least -1. Trying to apply the same argument at 0 would require evaluating the limit of $zf(z) = \cot(\pi z)/z$. The failure of that step alerts us to the existence of additional negative exponent terms in the power series at 0.

It seems to me that the key insight in the foregoing proof is using an integral to evaluate a sum. In this case, it is the machinery of residue calculus that connects the sum and integral. Once f has been defined, the remaining steps are a straightforward exercise of residue calculus methods. The next argument also uses an integral to evaluate a sum, and again involves complex numbers, but it has a distinctly different flavor. There, we use vector algebra techniques in the context of Fourier analysis.

Fourier Analysis. Before discussing the proof using Fourier analysis, it will be helpful to review a little vector analysis. In three-dimensional space, think of a vector as a directed line segment (that is, a segment with an arrow at one end). For vectors a and b, a fundamental operation is the *dot* or *inner* product $a \cdot b$. This may be defined as the product of the lengths of a and b and the cosine of the angle between them. Thus, if a and b are perpendicular, then $a \cdot b = 0$, while for parallel a and b, the dot product is just the product of the lengths (or the negative of the product if the vectors are parallel and oppositely directed).

The inner product is useful in breaking down vectors into simple pieces. Let e_x, e_y, and e_z be vectors of length 1 starting at the origin and pointing along the x, y, and z axes. Every other vector in space can be built up using sums and multiples of these three special vectors. A typical example would be something of the form $3e_x + 5e_y + 1.3e_z$. This is the vector which begins at the origin and ends at the point (3, 5, 1.3). Just as the three coefficients, 3, 5, and 1.3, completely determine the vector in this example, so any vector is uniquely determined by its three coefficients relative to the e vectors.

Notice that since any two of the e vectors are perpendicular, their dot product is 0. And the dot product of any of these vectors with itself is 1. These two properties, which are characteristic of an *orthonormal basis*, provide a simple way to compute the coefficients which describe any vector. Indeed, if we have $a = pe_x + qe_y + re_z$, then by taking the dot product of each side with e_x we find $p = a \cdot e_x$. Similar reasoning leads to $q = a \cdot e_y$ and $r = a \cdot e_z$. That is, the coefficient for each of the e vectors can be found by computing the dot product of a with that vector. As another consequence of orthonormality, observe that $a \cdot a = p^2 + q^2 + r^2$. The derivation of this identity,

$$\begin{aligned} a \cdot a &= (pe_x + qe_y + re_z) \cdot (pe_x + qe_y + re_z) \\ &= p^2 e_x \cdot e_x + q^2 e_y \cdot e_y + r^2 e_z \cdot e_z + 2pqe_x \cdot e_y + 2pre_x \cdot e_z + 2qre_y \cdot e_z \\ &= p^2 + q^2 + r^2, \end{aligned}$$

again uses the fact that the dot product of any of the e's with itself is 1, while the dot product between two different e's is 0.

In Fourier analysis, there is a wonderful analogy with the ideas of vectors, dot products, and orthonormality. In place of vectors we deal with complex-valued functions of a real variable. The dot product of two functions is defined using integrals: $f \cdot g = (1/2\pi)\int_{-\pi}^{\pi} f(t)\overline{g(t)}\,dt$ (the bar denotes complex conjugation). In place of the special vectors e_x, e_y, and e_z, we have the functions $1 = e^{0it}, e^{\pm it}, e^{\pm 2it}, e^{\pm 3it}, \ldots$. These form an orthonormal basis, and any well-behaved function f can be expressed using the basis functions in just the same way that vectors in space can be expressed in terms of the e vectors. As was the case for vectors, the coefficients for the basis functions are just dot products. Thus, if we write $f(t) = \cdots + a_{-2}e^{-2it} + a_{-1}e^{-it} + a_0 + a_1 e^{it} + a_2 e^{2it} + \cdots$, then

$$\begin{aligned} a_2 &= f \cdot e^{2it} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-2it}\,dt \end{aligned}$$

and similarly for all the other coefficients. Finally, in Fourier analysis there is an analog for the formula $a \cdot a = p^2 + q^2 + r^2$. Because the coefficients in the Fourier case can be complex numbers, it is their squared absolute values (not simply their squares) that must be summed, but otherwise the analogy is exact. Thus, we have the formula $f \cdot f = \cdots + |a_{-2}|^2 + |a_{-1}|^2 + |a_0|^2 + |a_1|^2 + |a_2|^2 + \cdots$. It is this last fact that we use to derive the value of Euler's sum.

Here is how it works. The function to use is $f(t) = t$. By direct calculation,

$$\begin{aligned} f \cdot f &= \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2\, dt \\ &= \frac{1}{2\pi}\frac{t^3}{3}\Big|_{-\pi}^{\pi} \\ &= \frac{\pi^2}{3}. \end{aligned}$$

Now we will compute $f \cdot f$ in terms of the coefficients a_k. As an example, let's calculate a_2.

$$\begin{aligned} a^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} te^{-2it}\, dt \\ &= \frac{1}{2\pi}\frac{2it + 1}{4}e^{-2it}\Big|_{-\pi}^{\pi} \\ &= \frac{i}{2}. \end{aligned}$$

The last step in this calculation takes advantage of the fact that $e^{2\pi i} = 1$. A similar calculation done with an arbitrary integer n in place of 2 discovers that $a_n = \pm i/n$ for all n except 0, and that $a_0 = 0$. Thus, for every n but 0, $|a_n| = 1/n$, and $\cdots + |a_{-2}|^2 + |a_{-1}|^2 + |a_0|^2 + |a_1|^2 + |a_2|^2 + \cdots$ is none other than Euler's sum written twice. This leads to $2E = \pi^2/3$, and dividing by 2 completes the proof.

Interlude: An Application of Euler's Result. Let's take a break from all these proofs, and consider an application. If an infinite sum of positive terms converges, it can be used to create a probability distribution. Just so for Euler's sum. Let $p_k = (6/\pi^2)(1/k^2)$. Then the p_k sum to 1, and can be regarded as a discrete probability distribution, with p_k the probability of the k^{th} outcome. Does this distribution actually have any use? As it turns out, it does. In fact, p_k is the probability that two randomly selected positive integers have greatest common divisor (GCD) equal to k. One must be a little careful about what is meant by randomly selecting an integer, for there is obviously no way to make all the positive integers equally likely and still have total probability 1. This is a technical point that can be put aside for the moment, in favor of a heuristic approach. To proceed, define q_k to be the probability that two randomly selected positive integers have GCD k. We show that $q_k = p_k$.

The GCD of integers a and b equals k if and only if two conditions hold. First, both integers must be multiples of k. Second, the GCD of a/k and b/k must be 1. Now the probability that two randomly selected integers are both multiples of k is $1/k^2$. The probability that $\text{GCD}(a/k, b/k) = 1$, given that a and b are multiples of k, is just the same as the unconditional probability that two positive integers have GCD 1, for as a and b range over the multiples of k, a/k and b/k range over the full set of positive integers. Combining the two preceding observations shows that $q_k = q_1(1/k^2)$. Since the q_k must sum to 1, we see that $q_1 = 6/\pi^2$, hence $q_k = p_k$, as asserted.

In retrospect, knowing the value of Euler's sum was a necessary step in determining the distribution of the GCD function. As an interesting consequence, we can now assert that a randomly generated fraction will be in lowest terms with probability $6/\pi^2$. I found these ideas in [**1**] (which comments on the technical point we set aside above) and [**13**].

Let us return now to our tour of proofs and examine a final derivation.

A Real Integral with an Imaginary Value. The final proof was published by Russell [**17**]. It begins with the definite integral

$$I = \int_0^{\pi/2} \ln(2\cos x)\, dx.$$

Now $2\cos x = e^{ix} + e^{-ix} = e^{ix}(1 + e^{-2ix})$. Therefore, $\ln(2\cos x) = \ln(e^{ix}) + \ln(1 + e^{-2ix}) = ix + \ln(1 + e^{-2ix})$. We make the substitution in the integral and arrive at

$$\begin{aligned} I &= \int_0^{\pi/2} ix \; + \ln(1 + e^{-2ix})\, dx \\ &= i\frac{\pi^2}{8} + \int_0^{\pi/2} \ln(1 + e^{-2ix})\, dx. \end{aligned} \tag{4}$$

The next step is to replace the logarithm with a power series, and integrate term by term. The power series expansion is [**9**, p. 401]

$$\ln(1 + x) = x - x^2/2 + x^3/3 - x^4/4 + \cdots$$

or, replacing x by e^{-2ix},

$$\ln(1 + e^{-2ix}) = e^{-2ix} - e^{-4ix}/2 + e^{-6ix}/3 - e^{-8ix}/4 + \cdots.$$

Integrate:

$$\begin{aligned} \int \ln(1 + e^{-2ix})\, dx &= \frac{e^{-2ix}}{-2i} - \frac{e^{-4ix}}{-2i \cdot 2^2} + \frac{e^{-6ix}}{-2i \cdot 3^2} - \frac{e^{-8ix}}{-2i \cdot 4^2} + \cdots \\ &= \frac{-1}{2i}\left(e^{-2ix} - \frac{e^{-4ix}}{2^2} + \frac{e^{-6ix}}{3^2} - \frac{e^{-8ix}}{4^2} + \cdots\right). \end{aligned}$$

This last expression is to be evaluated from 0 to $\pi/2$. That yields

$$\begin{aligned} &\int_0^{\pi/2} \ln(1 + e^{-2ix})\, dx \\ &\quad = \frac{-1}{2i}\left(e^{-i\pi} - 1 - \frac{e^{-2i\pi} - 1}{2^2} + \frac{e^{-3i\pi} - 1}{3^2} - \frac{e^{-4i\pi} - 1}{4^2} + \cdots\right). \end{aligned}$$

Now every exponential either evaluates to 1 (for even multiples of $i\pi$) or to -1 (for odd multiples). Therefore, half of the terms drop out, and the remaining terms are all fractions with a -2 in the numerator and an odd square in the denominator. Thus

$$\int_0^{\pi/2} \ln(1 + e^{-2ix})\, dx = \frac{1}{i}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right).$$

As we have seen before, the odd terms of Euler's sum add up to $3/4$ of the total. Combining this with the fact that $1/i = -i$, we conclude that

$$\int_0^{\pi/2} \ln(1 + e^{-2ix})\, dx = \frac{-3i}{4} E.$$

At this point, we must return to the integral we first considered. Substituting the expression just derived into (4), we obtain

$$I = i\left(\frac{\pi^2}{8} - \frac{3}{4} E\right).$$

But I is real, and it is equal to a pure imaginary. This forces both sides of the equation to vanish. Setting the right-hand side to 0 gives us the familiar conclusion $E = \pi^2/6$. Setting the left-hand side to 0 produces an added bonus:

$$\int_0^{\pi/2} \ln(\cos x)\, dx = -\frac{\pi}{2} \ln 2.$$

In this whirlwind of manipulations, there is probably nothing that would have disturbed Euler. In contrast, a modern student of mathematics would find reasons for skepticism at practically every step. First off, the original integral is improper, so we need to worry about convergence. Next, in order to use the natural logarithm for complex variables, we need to be sure that we can restrict the complex numbers to a suitable domain. (In this case, it is enough to observe that we never need to apply the logarithm to a negative real.) Thirdly, the power series for the natural logarithm converges within a circle of radius 1 centered at 0 in the complex plane. Unfortunately, for every x in the domain of integration, e^{-2ix} is on the boundary of this circle, so that we must be concerned about convergence of the power series, too. (For this step we may appeal directly to Theorem 3.44 of [**16**].) And finally, there is the term by term integration of the sum. In general terms, we handle these problems by starting in the middle and working our way out. The idea is to start with the series formulation of the integral, but let the upper limit be less than $\pi/2$. Then we can justify the term by term integration and take a limit to reach the upper limit of $\pi/2$, determining the value for the integral in the process. Working in the other direction, now that we know that the integral exists in the case of the power series formulation, we are justified in performing the manipulations that generate the integral that the argument above started with. This verifies that the original improper integral is indeed defined. For additional comments on justifying the steps in the proof, see [**17**].

Conclusion

We have seen a variety of proofs of Euler's result. It is interesting how wide a range of mathematical subjects appeared in these proofs. Euler's proof has the appearance of direct algebraic manipulation, but involves an unfounded assumption about the properties of power series. The first valid proof we considered works directly from the definition of convergent power series by providing bounds for partial sums of Euler's series. Two proofs each involve replacing the sum with a different operation. Thus, in residue calculus, a sum of residues is replaced by a complex line integral, while in Fourier analysis, a sum of squared coefficients is replaced by a dot product. And finally, two proofs use a technique of interchanging a sum and an integral to transform Euler's series into another form that can be summed directly. It should come as no surprise that there are still more proofs of Euler's formula (including a few by Euler himself). The interested reader is encouraged to consult the references for more approaches and additional references. Of historical interest is reference [**11**], the first edition of which appeared in 1921. In this encyclopedic work, several proofs of Euler's result can be found (see articles 136, 156, 189, 210) in the context of general procedures for manipulating and analyzing series expansions. The proof in article 210 is closely related to the Fourier analysis proof given above.

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