The Geometry of Rolling Curves

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Roll a closed convex curve along a line and follow the path of any chosen point on the curve. In the simplest case, the well-known cycloid is traced by a point on a rolling circle. In general, the set of (pointed) closed convex curves produces a wide variety of traced curves. Which curves are produced this way? Given a curve, can it be traced by rolling a (pointed closed convex) curve? If so, which one? In this paper, we give the necessary and sufficient conditions for traceability in terms of the normals to the curve and construct the curve to be rolled.

Suppose that the tracing point is allowed to be inside or outside of the rolling curve. Suppose further that the "line of roll" is replaced by a curve and that nonconvex curves are allowed to roll, i.e., by requiring that the point of contact move smoothly with no sliding (arclengths must agree) and the tangent lines agree at the contact point. In all cases, we solve the local inverse problem, as before, in terms of the normals to the curve.

The geometry of rolling curves has been studied extensively by mechanical engineers and others (see bibliography; Besant's book is the earliest systematic study) but their solution of the inverse problem is somewhat incomplete. We thank Dr. Rundell for suggesting this problem. All curves are plane curves. For simplicity, C^{∞} differentiability is assumed unless explicitly stated otherwise. This topic may be suitable for an honors calculus class.

1. Necessary Condition.

We begin with a simple case. Let C be a closed convex curve which can *roll* along a line L, i.e., the curvature of C is positive except possibly on a nowhere dense set and so there are no "straight sides." This condition guarantees that the point of contact s is well defined and behaves as the arclength parameter on both C and L (cf. §3). The tracing point P can be placed inside, on, or outside C. These are illustrated in Fig. 1 with a circle.

If P is regarded as the origin, then C can be described by polar coordinates (r, θ). If ψ is the angle between the tangent line and radial line of C, then P traces out the curve $\overline{C}$ (Fig. 2) given by

$$x = s - r\cos\psi \qquad y = r\sin\psi \tag{1}$$

We claim that the radial line is normal to $\overline{C}$. Using the well-known equation $\tan\psi = r d\theta / dr$, we calculate

$$\begin{aligned} ds &= \sqrt{dr^2 + r^2 d\theta^2} \\ &= \sqrt{1 + \left(\frac{r d\theta}{dr}\right)^2}\, dr \\ &= \sqrt{1 + \tan^2\psi}\, dr \\ &= \sec\psi\, dr, \end{aligned}$$

and so

$$\frac{dx}{d\psi} = \frac{ds}{d\psi} - \frac{dr}{d\psi}\cos\psi + r\sin\psi = \sin\psi\left(r + \tan\psi\frac{dr}{d\psi}\right)$$

and

$$\frac{dy}{d\psi} = \cos\psi\left(r + \tan\psi\frac{dr}{d\psi}\right).$$

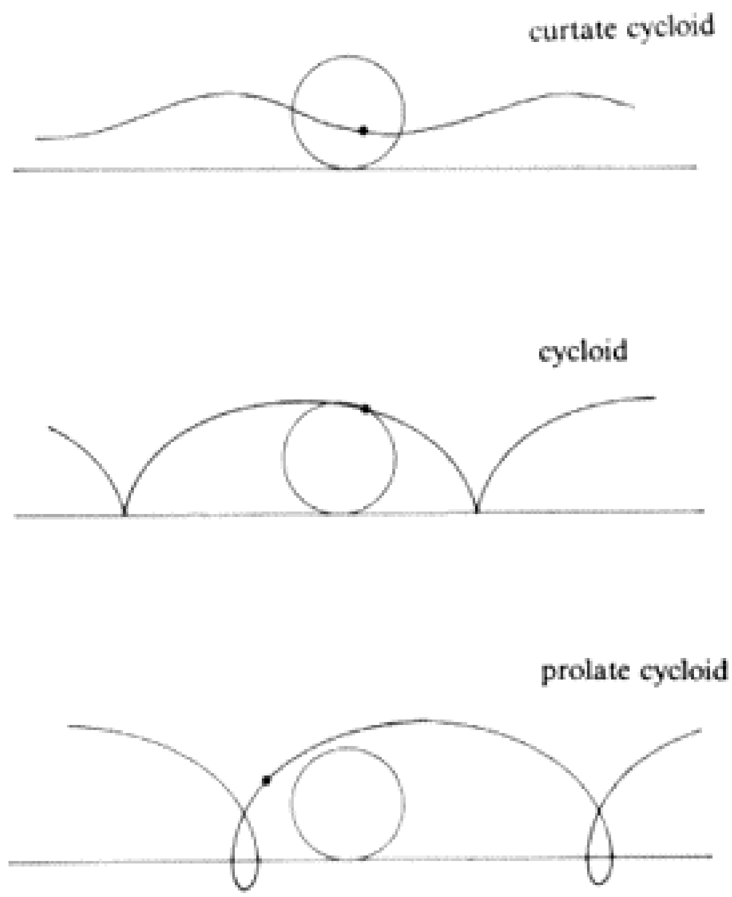

Figure 1

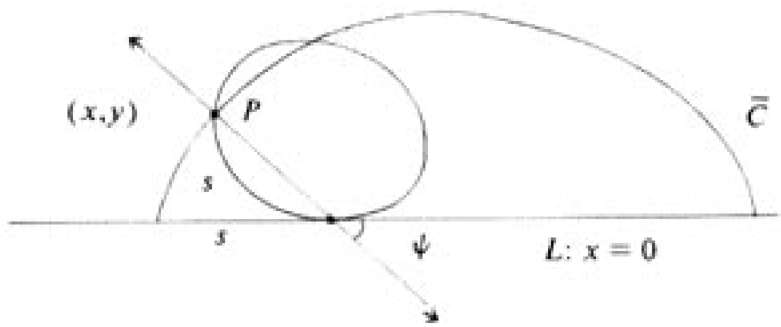

Figure 2

Hence $dy/dx = \cot\psi$ and the radial line is normal to $\overline{C}$. In particular,

CONDITION 1. The normals to $\overline{C}$ intersect L in increasing order.

This condition is sufficient for the local construction of C (Lemma 1).

An important observation is that the angle sum $\theta + \psi$ is the angle between the tangent lines of C and a fixed line (the polar axis). Since $d(\theta + \psi)/ds$ is the curvature of C by definition, the curvature assumption on C is equivalent to $d(\theta + \psi)/ds \geq 0$, with equality only on a nowhere dense set.

CONDITION 2. The function $(1/y)(dx/ds)$ is positive except possibly on a nowhere dense subset of $\overline{C}$.

It suffices to show

$$\frac{1}{y}\frac{dx}{ds} = \frac{d\theta}{ds} + \frac{d\psi}{ds}. \tag{2}$$

By construction,

$$\frac{y}{s - x} = \tan\psi \quad \text{and} \quad \frac{dy}{dx} = \frac{s - x}{y}.$$

So,

$$\begin{aligned}
\frac{d\psi}{dx} + \frac{d\theta}{dx} &= \frac{d}{dx}\left(\tan^{-1}\frac{y}{s - x}\right) + \frac{\tan\psi}{r}\frac{dr}{dx} \\
&= \frac{\frac{dy}{dx}(s - x) - y\left(\frac{ds}{dx} - 1\right)}{(s - x)^2 + y^2} + \frac{r\tan\psi}{r^2}\frac{dr}{dx} \\
&= \frac{\frac{(s - x)^2}{y} - y\sec\psi\frac{dr}{dx} + y + r\tan\psi\frac{dr}{dx}}{r^2}.
\end{aligned}$$

Since $y\sec\psi = r\tan\psi$, we obtain

$$\frac{d\psi}{dx} + \frac{d\theta}{dx} = \frac{1}{y}\left[\frac{(s - x)^2 + y^2}{r^2}\right] = \frac{1}{y},$$

which is equivalent to equation (2).

Equation (2) has several amusing interpretations. From the curvature assumption on C, it follows that P moves forward above L and backward below L. This is equivalent to the popular brain-teaser: What part of a train is moving backward? The answer is: the part of the inner wheel flange that drops below the track. Also, if $\overline{C}$ crosses L, then it crosses orthogonally, i.e., $\theta + \psi$ is defined everywhere and so $dx|_{\overline{C}} = 0$ whenever $y = 0$.

If nonconvex curves are rolled, then the position of P (above or below L) and the curvature of C at the contact point determine the direction P travels. It follows that the nonconvexity of C can be an obstruction to the smoothness of $\overline{C}$. For example, if C is nonconvex and if $\overline{C}$ lies above L, then $(1/y)\,dx$ changes sign and P moves forward and backward above L. If $\overline{C}$ is smooth, then it has a vertical tangent line. But the normal lines to $\overline{C}$ must intersect L, and hence $\overline{C}$ cannot be smooth.

2. *Sufficient Conditions*

Let $\overline{C}$ be a plane curve which satisfies condition 1 and which is periodic with respect to a line L. In this way, the arclength parameter s on L also parametrizes $\overline{C}$. The differentiability assumption on $\overline{C}$ is that the length $r(s)$ of the normal vector from $\overline{C}$ to L is smooth and $(dr/rds)\tan\psi$ (or equivalently $(1/y)(dx/ds) - (d\psi/ds)$) has a smooth extension over all its singularities. These smoothness conditions do not imply that $\overline{C}$ is smooth.

LEMMA 1. *If L and $\overline{C}$ are as above, then there is a smooth closed, not necessarily convex or simple, curve C and a distinguished point P which traces any finite piece of $\overline{C}$.*

Proof. Let r be the length of the normal vector from $\overline{C}$ to L, and let θ satisfy the differential equation

$$\frac{d\theta}{ds} = \frac{dr}{rds}\tan\psi.$$

The curve given by $(r(s), \theta(s))$ is smooth by assumption. To see that this curve is C, it suffices to check that the arclength parameter of $(r(s), \theta(s))$ is s.

Since equations (1) still hold, it follows that

$$\frac{dy}{d\psi} = \cos\psi\left(r + \tan\psi\frac{dr}{d\psi}\right) \quad \text{and} \quad \frac{dy}{dx} = \cot\psi.$$

Hence

$$\frac{dx}{d\psi} = \sin\psi\left(r + \tan\frac{dr}{d\psi}\right)$$

and, on differentiating equations (1),

$$\frac{ds}{d\psi} = \sec\psi\frac{dr}{d\psi}$$

$$= \sqrt{1 + \tan^2\psi}\frac{dr}{d\psi}$$

$$= \sqrt{1 + \left(\frac{rd\theta}{dr}\right)^2}\frac{dr}{d\psi}.$$

Hence $ds = \sqrt{dr^2 + (rd\theta)^2}$ and s is the arclength of C. If our construction of C does not result in a smooth closed curve, then we easily complete C to obtain the desired curve (Figs. 4 and 5). Q.E.D.

Figure 3

The local construction of C contains several peculiarities. For example, if all of $\overline{C}$ is used, and if $\overline{C}$ is the graph of $y = \log x$, $x \geq 1$, then C is a spiral. Another example is best illustrated by a curve $\overline{C}$ (solid curve) lying slightly above a cycloid (dotted curve) (Fig. 3). Our main theorem requires a period of $\overline{C}$, and it is possible for different periods to produce different curves C. Over a period like A, our construction of C produces a corner at the origin (solid curve) which is then completed (dotted curve) to a smooth closed curve (Fig. 4). Over a period like B, our construction of C produces a nonclosed curve which is then completed (Fig. 5). This dependence on the period can be resolved by introducing the following integral condition (3) into our main theorem.

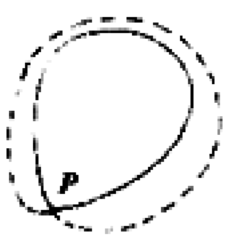

Figure 4

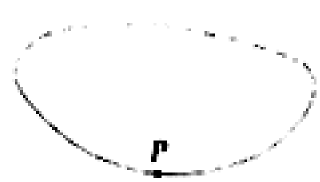

Figure 5

THEOREM 1. Let L and $\overline{C}$ be as in Lemma 1. Assume further that $\overline{C}$ satisfies Condition 2 and

$$\int_{\overline{C}_1} \frac{1}{y}\, dx = 2\pi \qquad (3)$$

where $\overline{C}_1$ is a period of $\overline{C}$ relative to L. Then there is a unique smooth closed convex curve C and a distinguished point P which traces all of $\overline{C}$.

Proof. At the endpoints of $\overline{C}_1$, the normals agree. Also, the change in ψ on $\overline{C}_1$, the integral condition (3), and equation (2) determine the change in θ. Namely, if ψ changes by 0, π, or 2π, then θ changes by 2π, π, or 0, respectively, where these correspond to, for example, the prolate, ordinary, and curtate cycloids in Fig. 1. If θ changes by 2π or 0, then applying Lemma 1 to $\overline{C}_1$ produces a closed curve C. If ψ changes by π then, by the periodicity of $\overline{C}$, r must be zero at the ends and also C is closed. The smoothness of C follows from the continuity (equivalently, closure) of C and the smoothness of the tangent vector $(dr/ds, d\theta/ds)$ to C. The convexity follows from Condition 2 and equation (2). Finally, C is unique up to a rotation of the (r, θ)-plane because r is uniquely determined by $\overline{C}$ and θ is defined up to a constant, or equivalently, a rotation.

Q.E.D.

3. Relaxation of Some Conditions

As mentioned earlier, a nonconvex curve C can roll under a suitable definition of “roll,” and the traced curve $\overline{C}$ will not be smooth. The nonsmoothness of $\overline{C}$ can also be caused by flat sides on C. Such sides force a discontinuity in the normal vector field to $\overline{C}$. If a square is rolled along a line and the tracing point P is chosen to be a corner, then we obtain Fig. 6. Here we clearly see the effects of corners and flat sides. When C rolls at a corner, the normals meet the line in a stationary fashion. The flat sides cause the singularities. Notice also that $\overline{C}$ is not convex, even though C is. If C is rounded slightly to produce a smooth convex curve, the traced curve $\overline{C}$ will still be nonconvex, although it will now be smooth.

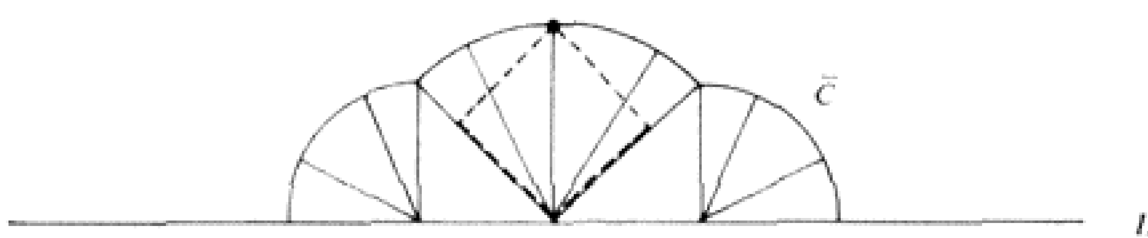

Figure 6

4. Rolling Curves Along Curves.

Let a curve C with tracing point P (not necessarily on C) be rolled along a curve $\underline{C}$ to produce a traced curve $\overline{C}$ (Fig. 7). We parametrize these curves by (r, θ), (z, u), and (x, y), respectively. Let α be the angle between the normal to $\underline{C}$ and the axis.

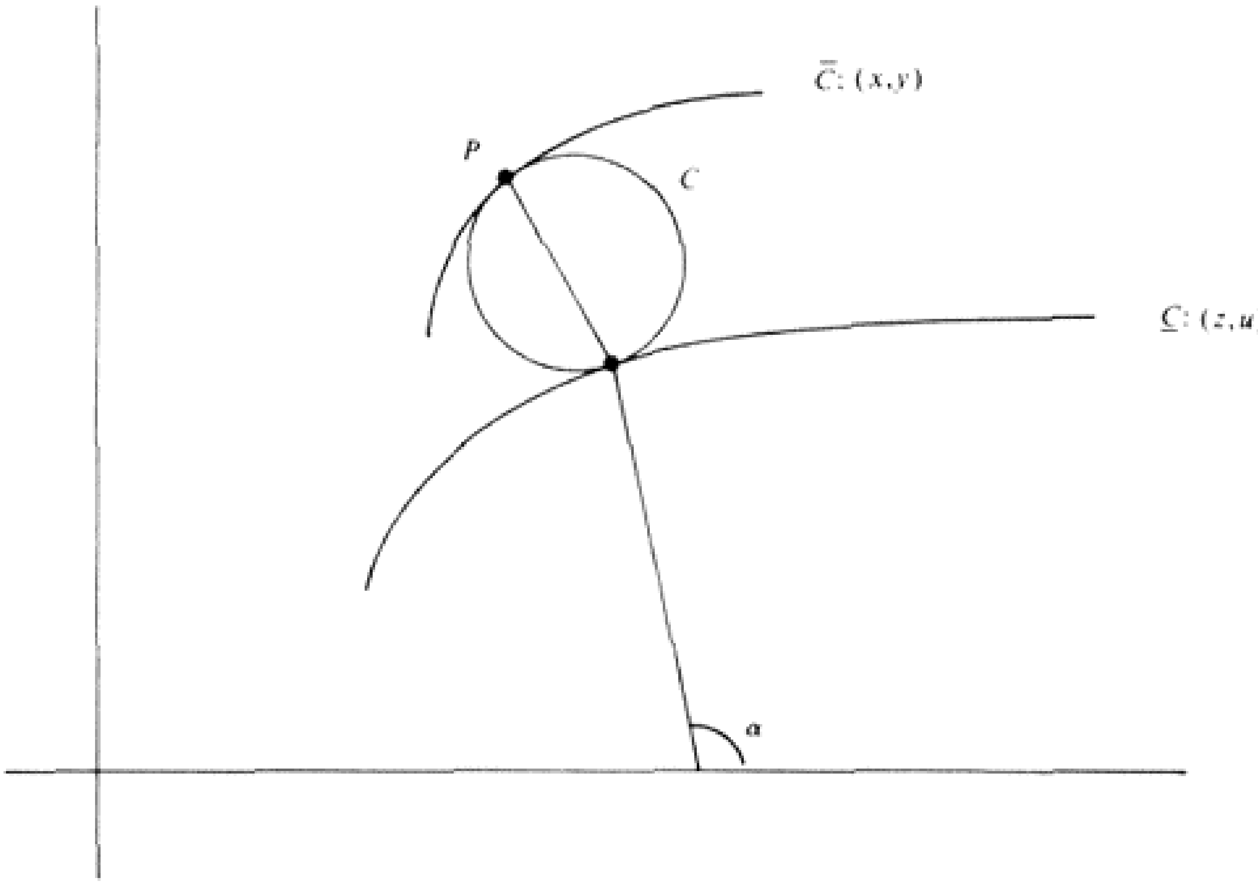

Figure 7

We claim that, as before, the line between P and the contact point is perpendicular to $\overline{C}$. Consider

$$x = r\cos(90^\circ + \alpha - \psi) + z = -r\sin(\alpha - \psi) + z \qquad (4)$$

$$y = r\sin(90^\circ + \alpha - \psi) + u = r\cos(\alpha - \psi) + u$$

where ψ is the angle between the normal to $\overline{C}$ and the tangent to $\underline{C}$ at the contact point. Since the arclength parameter on $\underline{C}$ agrees with that on C, we obtain

$$\sqrt{dz^2 + du^2} = ds = \sec\psi\, dr.$$

From $du/dz = \tan(90^\circ + \alpha)$ and $dz/du = -\tan\alpha$, it follows that

$$\sec\psi dr = -\sec\alpha\, du = \csc\alpha\, dz.$$

Differentiating equation (4),

$$\begin{aligned}\frac{dx}{d\alpha} &= -\frac{dr}{d\alpha}\sin(\alpha-\psi) - r\cos(\alpha-\psi)\left(1-\frac{d\psi}{d\alpha}\right) + \frac{dz}{d\alpha}\\ &= -\frac{dr}{d\alpha}\sin(\alpha-\psi) - r\cos(\alpha-\psi)\left(1-\frac{d\psi}{d\theta}\right) + \sin(\alpha-\psi+\psi)\sec\psi\frac{dr}{d\alpha}\\ &= \cos(\alpha-\psi)\left[-r\left(1-\frac{d\psi}{ds}\right)+\tan\psi\frac{dr}{d\alpha}\right].\end{aligned}$$

Similarly,

$$\frac{dy}{d\alpha} = \sin(\alpha-\psi)\left[-r\left(1-\frac{d\psi}{ds}\right)+\tan\psi\frac{dr}{d\alpha}\right],$$

and so

$$\frac{dy}{dx} = \tan(\alpha-\psi),$$

establishing perpendicularity.

Analogous to §2, the local construction of C proceeds by solving the differential equation

$$\frac{d\theta}{dr} = \frac{\tan x}{r},$$

where ψ and r are obtained from $\underline{C}$ and $\overline{C}$. To see that C is the desired curve, we work backward through the equations above to show that the arclength parameter on $\underline{C}$ agrees with that on C. Also, we need to calculate $d\psi + d\theta$, as before, for closure and convexity considerations.

$$\begin{aligned}\frac{d\psi}{dx}+\frac{d\theta}{dx} &= \frac{d}{dx}\left[\tan^{-1}\left(\frac{y-u}{z-x}\right)+\alpha+90^\circ\right]+\frac{\tan\psi}{r}\frac{dr}{dx}\\ &= \frac{\frac{d(y-u)}{dx}(z-x)-(y-u)\left(\frac{dz}{dx}-1\right)}{r^2}+\frac{d\alpha}{dx}+\frac{\tan\psi}{r}\frac{dr}{dx}\\ &= \frac{1}{r^2}\left[(z-x)\left(\tan(\alpha-\psi)+\frac{\cos\alpha}{\cos\psi}\frac{dr}{dx}\right)-(y-u)\left(\frac{\sin\alpha}{\cos\psi}\frac{dr}{dx}-1\right)+r^2\frac{d\alpha}{dx}+r\tan\psi\frac{dr}{dx}\right]\\ &= \frac{1}{r^2}\frac{dr}{dx}\left[(z-x)\frac{\cos\alpha}{\cos\psi}-(y-u)\frac{\sin\alpha}{\cos\psi}+r\tan\psi+r^2\frac{d\alpha}{dr}\right]+\frac{1}{r^2}\left[\frac{(z-x)^2}{y-u}+y-u\right]\\ &= \frac{1}{y-u}+\frac{1}{r}\frac{dr}{dx}\left[\frac{\sin(\alpha-\psi)\cos\alpha}{\cos\psi}-\frac{\cos(\alpha-\psi)\sin\alpha}{\cos\psi}+\tan\psi+r\frac{d\alpha}{dr}\right]\\ &= \frac{1}{y-u}+\frac{1}{r}\frac{dr}{dx}\left[r\frac{d\alpha}{dr}\right]\\ &= \frac{1}{y-u}+\frac{d\alpha}{dx}.\end{aligned}$$

Hence, C is convex if and only if

$$\frac{1}{y-u} + \frac{d\alpha}{ds} \geq 0.$$

The integral condition for closure is

$$\int \frac{1}{y-u}\,dx = 2\pi - \Delta\alpha$$

where the integral and the change in α are taken over one period. Note that y and u are the "heights" of the curves $\overline{C}$ and $\underline{C}$, but not necessarily above the same point. A point (x, y) lies above (z, u) only when $\alpha - \psi = 90°$ or, equivalently, the normal to (x, y) is vertical.

Previous considerations carry through with the obvious changes. Convexity corresponds to the curvature of $C(s)$ being greater than the curvature of $\underline{C}(s)$. Flat sides correspond to congruent pieces of C and $\underline{C}$ which roll against each other.

Finally, one can ask the general question: Given $\overline{C}$, is there a pair $(C, \underline{C})$ of curves so that C rolls on $\underline{C}$ to produce $\overline{C}$. If no smoothness is required, the answer is yes, but some curves $\overline{C}$ admit no pairs C, $\underline{C}$ with the curvature of $C(s)$ greater than the curvature of $\underline{C}(s)$, C, $\underline{C}$ smooth. Details are left to the reader.

References

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