

## Tape Counters

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The tape counter on many VCRs and audiocassette players is an example of a function, a practical function that at first may seem mysterious. Anyone who has played a VCR probably has noticed that the counter reading is not a simple linear function of the time. For example, the following data came from reading a 6 hour tape at one hour intervals:

<u>Time (minutes)</u>	<u>Counter</u>
60	1540
120	2669
180	3604
240	4422
300	5157
360	5831

If we let  $f(t)$  denote the counter reading as a function of time we see that  $f$  is an increasing function, but its rate of increase slows down (that is,  $f''(t)$  is negative). It's not obvious what kind of function  $f(t)$  is. When the VCR operates, the tape moves past the heads at a constant speed  $k$ . As it is wound onto the take-up reel, the radius increases, and hence the reel turns more slowly. The number on the tape counter is proportional to the number of turns of the take-up reel. (Some of the newest models of VCR, however, have now replaced this kind of tape counter with one that gives the time elapsed.)

How do we determine the function  $f(t)$ ? At a given time  $t$ , let  $s$  denote the length of tape which has been wound,  $r$  denote the radius of the tape on the take-up reel,  $n$  the number of turns the take-up reel has made, and  $\theta$  denote the angle (in radians) through which the reel has turned. If the initial radius of the tape on the take-up reel is  $r_0$  and the thickness of the tape is  $b$ , then it is easy to see that  $\theta = 2\pi n$  and  $r = r_0 + nb$ . See Figure 1.

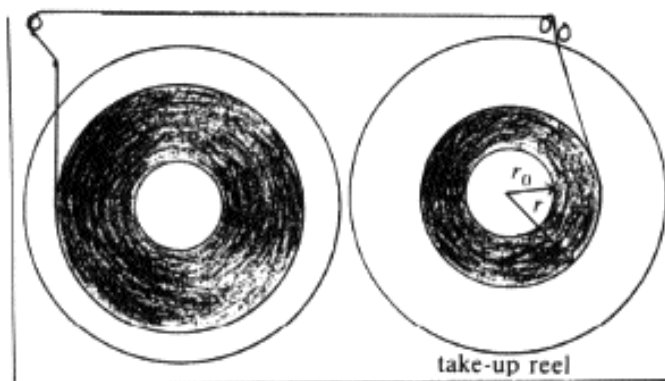


Figure 1. Side view of a videocassette.

We make the assumption that  $b$  is very small compared to  $r$  at any time, and hence that a winding of the tape may be approximated by a circle of radius  $r$ . Then for a small rotation  $\Delta\theta$ , the length of the tape wound is  $\Delta s = r\Delta\theta$ . Hence

$$s = \int_0^\theta r d\theta = \int_0^n (r_0 + nb)2\pi dn \quad (1)$$

and we see that

$$s = \pi bn^2 + 2\pi r_0 n. \quad (2)$$

Formula (2) for the tape length  $s$  is of course applicable in any problem of winding tape, rope, or ribbon on a spool or roll where the thickness of the tape is small relative to the radius of the spool. It isn't necessary to use calculus to derive formula (2) and on the other hand, one can also derive an exact (but more complicated) formula for  $s$ ; see Box 1.

### ***Finding a Formula for $s$***

Formula (2) can also be derived easily without calculus using the following geometric approach. A side view of the reel shows that the area of the tape wound on the reel is that of a "washer" of outer radius  $r$  and inner radius  $r_0$ , hence equals  $\pi r^2 - \pi r_0^2$ . But this same area also equals the length of the wound tape  $s$  times the tape thickness  $b$ .

Thus

$$\begin{aligned} sb &= \pi r^2 - \pi r_0^2 = \pi[(r_0 + nb)^2 - r_0^2] \\ &= 2\pi r_0 nb + \pi n^2 b^2 \end{aligned}$$

Dividing by  $b$  yields equation (2).

One can obtain the exact value of  $s$  by using polar coordinates to study the curve. We have  $r = r_0 + nb = r_0 + (b/2\pi)\theta = g(\theta)$  as the equation in polar coordinates. Since the formula for arc length in polar coordinates is

$$ds = \sqrt{[g(\theta)]^2 + [g'(\theta)]^2} d\theta,$$

$s$  is given by the formula

$$s = \int_0^\theta \sqrt{r^2 + \left(\frac{b}{2\pi}\right)^2} d\theta. \quad (3)$$

If we drop the term  $(b/2\pi)^2$  in equation (3) (since  $b$  is very small compared to  $r$ ), and simplify, we get the equation  $s = \int_0^\theta r d\theta$  which was used in equation (1). It is possible to evaluate the integral in (3) by standard techniques, but the resulting function would be much more complicated than that described in equation (2). [Note that the informal geometric proof of (2) given above is justified by assuming that the wound tape consists of concentric circles instead of being a spiral.]

Now in a VCR, the tape moves at a constant speed  $k$  so we know  $s = kt$  for some constant  $k$ . The counter reading  $m$  is a constant multiple of the number of turns  $n$ ; that is  $m = cn$  for some constant  $c$ . (I have found VCRs where apparently  $c = 2$  or  $4$ , for example.) Substituting into equation (2) yields

$$kt = \pi b \frac{m^2}{c^2} + 2\pi r_0 \frac{m}{c}$$

and hence

$$t = \left[ \frac{\pi b}{c^2 k} \right] m^2 + \left[ \frac{2\pi r_0}{ck} \right] m = Am^2 + Bm \quad (4)$$

which is a quadratic function whose graph is part of a parabola passing through the origin. To find the function  $f(t)$  (the counter function), we simply invert the function described in (4) to get

$$m = f(t) = \frac{-B + \sqrt{B^2 + 4At}}{2A} \quad (5)$$

Thus the function  $f(t)$  is a modified square root function and its graph is the upper part of a parabola opening to the right and passing through the origin. It's a naturally occurring inverse function, something that should interest our students.

Since it's hard to get accurate values for the constants such as  $b$  and  $r_0$ , the easiest way to calculate  $A$  and  $B$  is simply to use two test values in equation (4) and solve the simultaneous equations for  $A$  and  $B$ . For example using  $t = 60$ ,  $m = 1540$  and  $t = 240$ ,  $m = 4422$  yields  $A = 5.31334E - 06$  and  $B = 3.07785E - 02$ . I used these values in equation (5), and checking minute by minute, found that the formula for  $m$  matched the readings with a discrepancy of at most  $\pm 2$ . I have also found that different tapes give different readings, even when they are the same brand and type. (For example when  $t = 240$ , besides the reading of 4422 on the tape described in this example, I have found readings  $m = 4310$  and  $m = 4370$  on different tapes.)

What happens if the tape is being wound onto a reel which is turning at a constant speed? Substitute  $n = kt$  into equation (2). You'll see that the tape length is now growing as a quadratic function of time.

Using formula (4) or (5) you can generate a handy reference table for use with your VCR. What happens, however, if someone resets the counter in the middle of your tape? Or if you start a tape that has been played part way through? The table can't be used, but you can still estimate how far the tape has been played by using the derivative  $dm/dt$ .

Differentiation of equation (5) yields

$$\frac{dm}{dt} = \frac{1}{\sqrt{B^2 + 4At}} \quad (6)$$

Solving for  $t$  gives

$$t = \frac{1}{4A} \left( \frac{1}{(dm/dt)^2} - B^2 \right) \quad (7)$$

In addition if we differentiate equation (4) with respect to  $m$  and use  $dm/dt = 1(dt/dm)$  we find

$$\frac{dm}{dt} = \frac{1}{2mA + B} \quad (8)$$

which can also be inverted to express  $m$  in terms of  $dm/dt$ .

If we can estimate  $dm/dt$  then we can use these formulas to estimate  $t$  and  $m$ . A simple-minded way to get a rough estimate for  $dm/dt$  is to run the VCR for one minute ( $\Delta t = 1$ ) and calculate  $\Delta m$  from the counter. There are also more sophisticated methods which involve getting several values of the function and then using elementary numerical analysis to estimate the derivative.

Be warned, however:  $dm/dt$  changes rapidly at the beginning of the tape, and much more slowly at the end. For the particular tape used in this example, we computed the following data from equation (7).

$dm/dt$	$t$
30	8
26	15
22	53
18	101
14	195
13	234
12	282
11	344

It follows that near the end of the tape, where  $dm/dt$  changes slowly, we need to know it more accurately in order to approximate  $t$  or  $m$ . We might let the tape run for ten minutes, for example, and divide  $\Delta m$  by 10 to get an estimate with one decimal place. A stopwatch might also be used for more careful estimates.

**Addendum.** It has come to my attention that some of the material in this paper has previously appeared in an article by Arnold J. Insel: "Cassette Tape: Predicting Recording Time," the *UMAP Journal*, Vol. V, No. 2, 1984, pp. 200–214.

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