Tooth Tables: Solution of a Dental Problem by Vector Algebra

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The following problem in dentistry concerning the way in which a tooth is prepared for a gold inlay was posed to me by two instructors at the University of Nebraska Dental College, Linda Dubois and Stanley Kull. They expected that a mathematical analysis of the problem would require heavy use of a computer. However, the problem they posed is nicely amenable to formulation (and solution) using some elementary facts of three-dimensional vector algebra which are usually included in the standard first course in calculus. The solution we obtained and its implications for dentistry have been described in the paper **[2]**. But, since **[2]** was written for dentists, the mathematics was only briefly summarized and little attempt was made to explain the details of the mathematical formulation and solution of the problem. Consequently, it seemed appropriate to write this paper in order to give a full account of the mathematics involved.

In order to describe the problem we will need a small amount of dental terminology which can be easily learned from FIGURE 1 and the accompanying "Dental Vocabulary" gleaned from Dorland's Medical Dictionary **[1]** and Gray's Anatomy **[3]**. A tooth is prepared for a gold inlay by carving out (from the occlusal surface and from a proximal surface of contact with a neighboring tooth) a cavity shaped more or less as illustrated in FIGURE 1. This cavity is called an **inlay preparation.**

The **proximal box** is that part of the inlay preparation which is bounded in the rear by the **axial wall,** on the sides by the left and right walls, on the bottom by the **seat,** on the top by a portion of the **occlusal** (which has been cut away), and on the front by a portion of the **proximal surface of contact** (which has also been cut away).

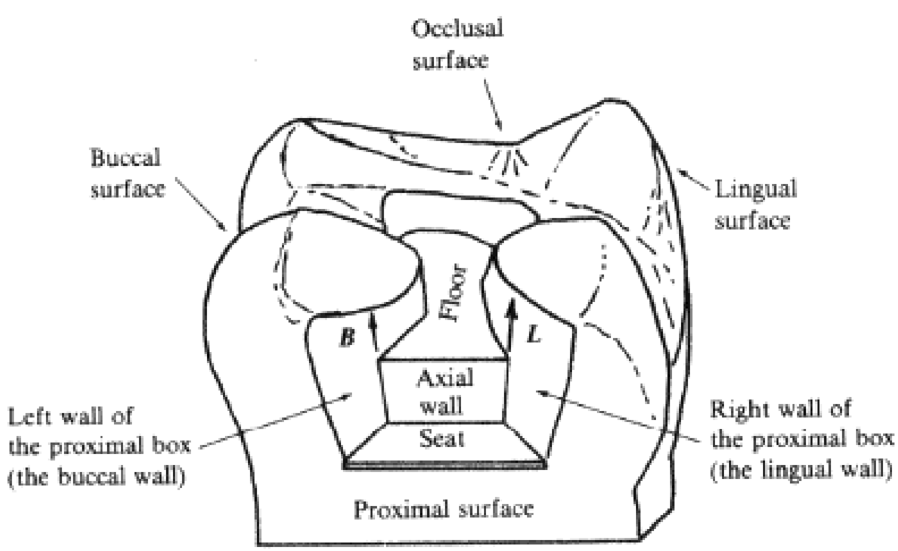

FIGURE 1. A tooth (molar) showing a gold-inlay preparation carved out of it.

A Brief Dental Vocabulary
Buccal surface is the cheek side of the tooth. **Lingual** surface is the tongue side of the tooth. **Occlusal** surface is the chewing surface of the tooth. **Preparation (**or **inlay preparation**) is the shaped cavity cut out of a tooth to receive a gold inlay. **Proximal** is next to the neighboring tooth. **Proximal box** is that portion of the inlay preparation nearest to the neighboring tooth. **Proximal surface** is the surface of the tooth which is in contact with the neighboring tooth. **Seat** (or **gingival seat**) is the seat of the proximal box nearest the gums. **Floor** of inlay preparation is parallel to the seat but is closer to the occlusal surface. It is more-or-less perpendicular to the axial wall and extends from the axial wall to the center of the tooth. The problem discussed in this paper is not concerned with the part of the inlay preparation which lies between the floor and the occlusal surface.

It is required to cut the cavity for the gold inlay in such a manner that a wax mold of the cavity can be drawn out toward the occlusal and the gold inlay itself can be dropped into the cavity from the occlusal side of the tooth. This is called an **occlusal draw.** In order to guarantee a good occlusal draw it is both necessary and sufficient that all of the nearly-vertical walls of the inlay preparation should be tilted slightly back so that they all face slightly toward the occlusal surface of the tooth.

The lines of intersection of the axial wall with the left and right walls of the proximal box will be called the **axial corner lines.** Their continuations are marked in FIGURE 1 by arrows **B** and **L** (for "buccal" and "lingual"). Since these axial corner lines lie in a common plane (namely, the plane of the axial wall) they are either parallel or else they intersect. If they intersect on the occlusal side of the seat, then we say that these lines "converge in the direction of the occlusal" or simply "**converge**." On the other hand, if they intersect on the root side of the seat, then we say that these lines "diverge in the direction of the occlusal" or simply "**diverge**."

It has been customary for dentists to cut inlay preparations in such a way that the axial corner lines diverge in the occlusal direction as illustrated by the two arrows **B** and **L** shown in FIGURE 1. Evidently, this divergence of **B** and **L** has been generally regarded by dentists (see **[2]** and the references cited there) as both necessary and sufficient in order to avoid an undercut by the proximal box walls in an inlay preparation. Indeed, divergence is frequently used as a visual criterion for a good occlusal draw. It is the main purpose of this paper to prove by means of a mathematically rigorous argument that this divergence is neither necessary nor sufficient for an occlusal draw. In fact, we show that

(i) *Convergence of* **B** *and* **L** *toward the occlusal can be consistent with a good occlusal draw;*

(ii) *Divergence is no guarantee of a good occlusal draw.*

One consequence of this result which may have important significance for dentistry is that when an inlay preparation is cut with *convergence* (consistent with draw) it *allows less good tooth material to be cut away* than when one requires divergence.

It remains of course for dental instructors to perfect a procedure which, on the one hand, allows convergence and guarantees a good draw and, on the other hand, is simple enough to be practical for dentists to learn and to be able to put into practice. In this paper we have merely shown that it is geometrically possible.

The mathematical formulation and solution of the problem

We are concerned with the geometry of the proximal box, and it suffices to consider the right-hand side since the left-hand side could be treated in a completely analogous manner. We introduce a three-dimensional rectangular coordinate system oriented so that the seat of the proximal box lies in the *xy*-plane, the *z*-axis coincides with the central axis of the tooth (directed from the root out through the occlusal surface in the direction of draw), the *y*-axis is parallel to the plane of the axial wall and passes out through the right (lingual) surface, and the *x*-axis pierces at right angles the line of intersection of the axial wall with the seat and then exits the tooth through the proximal surface. This rectangular coordinate system is illustrated in FIGURE 2. Instead of comparing the right **L** and left **B** axial corner lines (for convergence or divergence toward the occlusal), we will compare the right (or lingual) axial corner line **L** with the line of intersection of the axial wall with the *xz*-plane. This latter line we call the **"medial" line.**

As usual, we use the letters **i, j, k** to denote the unit vectors pointing in the *x, y, z* directions, respectively. Recall that the orientation of a plane in three-dimensional space can be conveniently described mathematically by means of a unit normal vector (a unit vector perpendicular to the surface of the plane). In FIGURE 2, the vector labeled **A** is the unit normal vector to the plane of the axial wall, and the vector labeled **P** is the unit normal to the plane of the right (lingual) wall of the proximal box. Both of these vectors are chosen so that their directions point *into* the proximal box. Also in FIGURE 2, the vector labeled **M** is parallel to the medial line and the vector labeled **L** is parallel to the right (lingual) axial corner line.

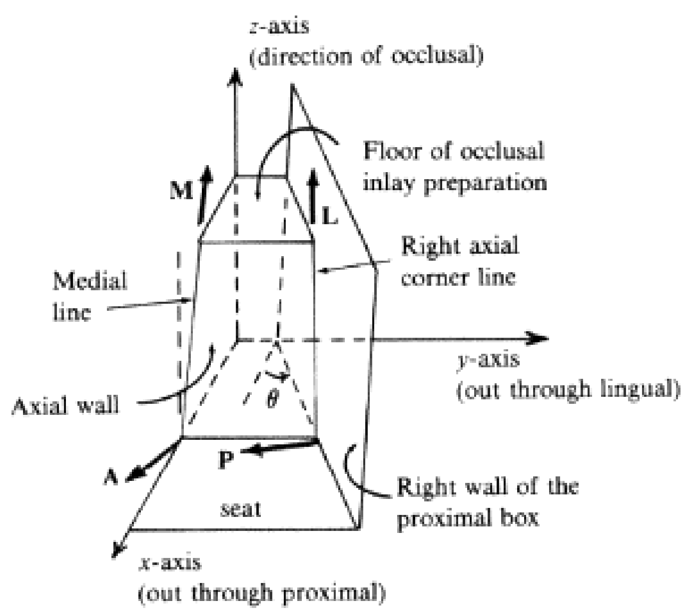

FIGURE 2. An appropriate rectangular coordinate system.

For the formulation and solution of our problem we shall need to use the following well-known properties of the "cross" and "dot" products of the two vectors **U** and **V**, which can be found in any good calculus text.

(1) Definitions. If $\mathbf{U} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{V} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then the **cross product** of **U** and **V** is defined as

$$\mathbf{U} \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix},$$

or

$$\mathbf{U} \times \mathbf{V} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

The **dot product** of **U** and **V** is defined as $\mathbf{U} \cdot \mathbf{V} = u_1v_1 + u_2v_2 + u_3v_3$. The **length** $|\mathbf{U}|$ is defined to be $[(u_1)^2 + (u_2)^2 + (u_3)^2]^{1/2}$.

(2) $\mathbf{U} \times \mathbf{V}$ is perpendicular to both **U** and **V**.

(3) $\mathbf{U} \times \mathbf{V} = -\mathbf{V} \times \mathbf{U}$.

(4) The "right-hand rule." If $0 \le \phi \le \pi$ is the (smaller) angle between **U** and **V**, then $\mathbf{U} \times \mathbf{V}$ points in the direction of the thumb when the right-hand fingers are folded (through the angle ϕ) from the direction of **U** into the direction of **V**.

(5) $|\mathbf{U} \times \mathbf{V}| = |\mathbf{U}|\,|\mathbf{V}| \sin \phi$.

(6) $\mathbf{U} \cdot \mathbf{V} = |\mathbf{U}|\,|\mathbf{V}| \cos \phi$.

(7) $\mathbf{U} \times (\mathbf{V} \times \mathbf{W}) = (\mathbf{U} \cdot \mathbf{W})\mathbf{V} - (\mathbf{U} \cdot \mathbf{V})\mathbf{W}$.

(8) $\mathbf{U} \cdot (\mathbf{V} \times \mathbf{W}) = (\mathbf{U} \times \mathbf{V}) \cdot \mathbf{W} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$

If this product is positive (negative), the triple **U**, **V**, **W** is called dextral (sinistral).

Now note that since **M** is parallel to the *xz*-plane and also to the axial plane (see FIGURE 2), **M** must be perpendicular to both of the vectors **j** and **A**. Therefore (since the length of **M** is immaterial), we may write

$$\mathbf{M} = \mathbf{A} \times \mathbf{j} \tag{9}$$

and similarly,

$$\mathbf{L} = \mathbf{P} \times \mathbf{A}. \tag{10}$$

Since **M** and **L** are both parallel to the axial wall, it follows that either $\mathbf{M} \times \mathbf{L} = \mathbf{0}$ (when they are parallel to each other) or $\mathbf{M} \times \mathbf{L}$ is a nonzero scalar multiple of the vector **A**. Consideration of the right-hand rule (4) and the algebraic sign of this scalar coefficient of **A** yields the following criteria:

I. *The right (lingual) corner line converges toward the medial line in the direction of the occlusal precisely when*

$$\mathbf{M} \times \mathbf{L} = \lambda \mathbf{A}, \text{ with } \lambda > 0.$$

II. *The right (lingual) corner line diverges from the medial line in the direction of the occlusal precisely when*

$$\mathbf{M} \times \mathbf{L} = -\lambda \mathbf{A}, \text{ with } \lambda > 0.$$

III *The condition for a good occlusal draw is met (as far as the proximal box is concerned) precisely when the normal vectors* **A** *and* **P** *(as well as the normal vector to the left wall) each have a positive z-component.*

The question then is this: Is occlusal convergence of **L** toward **M** consistent with a good occlusal draw? That is, can conditions I and III be simultaneously met, rather than (as traditionally believed by dentists) only II and III?

In order to answer this question, we will need the *x, y,* and *z* components of the unit normal vectors **A** and **P.** These can be obtained from FIGURES 3, 4, and 5 which are two-dimensional figures extracted from FIGURE 2.

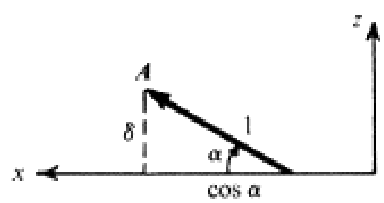

FIGURE 3. Components of the vector A; here $\delta = \sin \alpha$**.**

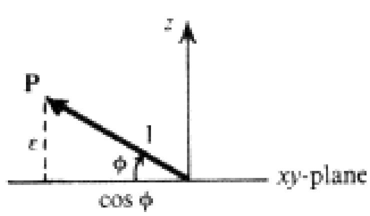

FIGURE 4. The ***z*****-component** ε **of the vector P; here** $\varepsilon = \sin \phi$**.**

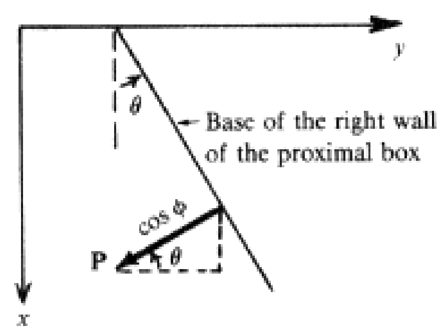

FIGURE 5. Projection of P onto the ***xy*****-plane.**

From FIGURE 3 we see that

$$\mathbf{A} = (\cos\alpha)\mathbf{i} + (\sin\alpha)\mathbf{k} \tag{11}$$

and from FIGURES 4 and 5 we see that

$$\mathbf{P} = (\cos\phi\sin\theta)\mathbf{i} - (\cos\phi\cos\theta)\mathbf{j} + (\sin\phi)\mathbf{k}. \tag{12}$$

Extracting the z-components from (11) and (12), criterion III says that the condition for good occlusal draw is met when $\delta = \sin\alpha > 0$ and $\varepsilon = \sin\phi > 0$. The axial plane will be undercut if $\delta < 0$, and the right wall of the proximal box will be undercut if $\varepsilon < 0$.

We can apply some of the properties of cross product to rewrite $\mathbf{M} \times \mathbf{L}$ in criteria I and II:

$$\begin{aligned}\mathbf{M} \times \mathbf{L} &= \mathbf{M} \times (\mathbf{P} \times \mathbf{A})\\ &= (\mathbf{M}\cdot\mathbf{A})\mathbf{P} - (\mathbf{M}\cdot\mathbf{P})\mathbf{A} = -(\mathbf{M}\cdot\mathbf{P})\mathbf{A}.\end{aligned}$$

Thus criteria I and II can be rephrased as follows:

I′, II′. $\mathbf{M}\cdot\mathbf{P}$ *should be negative for occlusal convergence and positive for occlusal divergence.*

But by formulas (8), (9), (11), and (12) we obtain

$$\begin{aligned}\mathbf{M}\cdot\mathbf{P} &= (\mathbf{A}\times\mathbf{j})\cdot\mathbf{P}\\ &= \begin{vmatrix}\cos\alpha & 0 & \sin\alpha\\ 0 & 1 & 0\\ \cos\phi\sin\theta & -\cos\phi\sin\theta & \sin\phi\end{vmatrix}\\ &= \cos\alpha\sin\phi - \sin\alpha\cos\phi\sin\theta.\end{aligned}$$

Therefore, the criterion for occlusal convergence of the right axial corner line $\mathbf{L}$ toward the medial line $\mathbf{M}$ becomes

$$\tan\phi < \tan\alpha\sin\theta, \tag{13}$$

while the criterion for a good occlusal draw may be expressed as

$$\tan\phi > 0 \text{ and } \tan\alpha > 0, \tag{14}$$

because $\cos\phi$ and $\cos\alpha$ are positive in the realistic range $|\phi| < \pi/2$ and $|\alpha| < \pi/2$.

Conditions (13) and (14) may easily hold simultaneously for a continuum of values of α, ϕ, and θ. That is, *convergence is definitely consistent with a good occlusal draw.*

The reverse inequalities

$$\tan\phi > \tan\alpha\sin\theta, \tag{15}$$

and

$$\tan\phi < 0 \text{ and/or } \tan\alpha < 0, \tag{16}$$

are also easily satisfied simultaneously for a continuum of values of ϕ, α, and θ with $0 < \theta < \pi/2$. But (15) is the condition for *occlusal divergence* of the right axial corner line $\mathbf{L}$ away from the medial line $\mathbf{M}$, and (16) is the condition for failure of occlusal draw, because of an undercut of either the right proximal wall ($\tan \phi < 0$) or the axial wall ($\tan \alpha < 0$). Of course (15) implies that $\tan \phi$ can be negative only if $\tan \alpha$ is also negative, because $\sin \theta$ is always positive ($0 < \theta < \pi/2$). Thus *occlusal divergence is not a guarantee of a good occlusal draw*, and thus should not be used as a visual test for such. Note that for simultaneous convergence and draw it must be true that $\tan \phi < \tan \alpha$ because $0 < \sin \theta < 1$.

The criterion (I′) that $\mathbf{M} \cdot \mathbf{P}$ should be negative for convergence has important geometric implications. The dot product $\mathbf{M} \cdot \mathbf{P}$ negative means that the component of $\mathbf{P}$ in the direction of $\mathbf{M}$ is negative, i.e., the projection of $\mathbf{P}$ along the line through $\mathbf{M}$ has a direction opposite to that of $\mathbf{M}$. In addition, $\mathbf{M} \cdot \mathbf{P} = (\mathbf{A} \times \mathbf{j}) \cdot \mathbf{P}$ is *negative* if and only if the three vectors $\{\mathbf{A}, \mathbf{j}, \mathbf{P}\}$ form a *left-handed* (sinistral) coordinate system.

A table of values for the angles that allow simultaneous convergence and draw

In addition to the angles α, ϕ, and θ introduced in FIGURES 3, 4, and 5, we can introduce the angle μ between $\mathbf{M}$ and $\mathbf{L}$ and call it the "angle of convergence" (taken positive for convergence and negative for divergence). We defined $\mathbf{M}$ in (9) as $\mathbf{M} = \mathbf{A} \times \mathbf{j}$; using equation (11), we find that

$$\mathbf{M} = \mathbf{A} \times \mathbf{j} = (-\sin \alpha)\mathbf{i} + (\cos \alpha)\mathbf{k},$$

so that $\mathbf{M}$ is a unit vector. From (6) it follows that $\mathbf{M} \cdot \mathbf{L} = |\mathbf{L}| \cos \mu$. Also from (11) and (12) we have

$$\mathbf{M} \cdot \mathbf{L} = (\mathbf{A} \times \mathbf{j}) \cdot (\mathbf{P} \times \mathbf{A}) = \cos \phi \cos \theta,$$

so that

$$\cos \mu = (\cos \phi \cos \theta)/|\mathbf{L}|. \tag{17}$$

Properties (3), (5), and (8) give

$$\mathbf{M} \times \mathbf{L} = |\mathbf{L}|\mathbf{A} \sin \mu = -(\mathbf{M} \cdot \mathbf{P})\mathbf{A},$$

so that

$$\sin \mu = (\sin \alpha \cos \phi \sin \theta - \cos \alpha \sin \phi)/|\mathbf{L}|. \tag{18}$$

Combining (17) and (18) gives

$$\tan \mu = \sin \alpha \tan \theta - \cos \alpha \tan \phi \sec \theta. \tag{19}$$

Thus the angle of convergence μ is completely determined by the angles α, ϕ, and θ.

By using standard tables of the trigonometric functions (such as those found in [**4**]), or a calculator with trig function capability, one can easily construct a table of acceptable values of the angles α, ϕ, and θ and compute the corresponding values of μ by means of the formula (19). We give an example of such a table in TABLE 1.

α	ϕ	θ	μ
6°	3°	30°	0.01°
"	"	35°	0.55°
"	"	40°	1.13°
"	"	45°	1.77°
"	"	50°	2.49°
"	"	55°	3.34°
"	"	60°	4.39°
"	"	65°	5.76°
"	"	70°	7.68°

α	ϕ	θ	μ
7°	3°	26°	0.09°
"	"	30°	0.59°
"	"	35°	1.25°
"	"	40°	1.97°
"	"	45°	2.77°
"	"	50°	3.68°
"	"	55°	4.77°
"	"	60°	6.11°
"	"	65°	7.87°
"	"	70°	10.36°

Table 1

α = angle of tilt of axial wall back from vertical.

ϕ = angle of tilt of proximal wall back from vertical.

θ = angle of (tangent plane) of right wall of proximal box.

μ = angle of convergence between vectors **M** and **L**.

$$0 < \tan\phi < \tan\alpha \sin\theta$$

Draw ↑ ↑ Convergence

I would like to thank Paul Campbell for encouraging me to expand and write up this problem, which was the topic of a talk I presented to the Nebraska-South Dakota Section of the MAA at its meeting in Vermillion, South Dakota in April of 1981.

References

[1] Dorland's Illustrated Medical Dictionary, Twenty-fifth ed., W. B. Saunders, Philadelphia, 1974.

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[3] Henry Gray, Anatomy of the Human Body, Twenty-ninth American Edition, C. M. Goss, ed., Lea & Febiger, Philadelphia, 1975.

[4] S. M. Selby, ed., Standard Mathematical Tables, Twenty-first ed., The Chemical Rubber Co., Cleveland, 1973.