Lagrange Multiplier Problems in Economics

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1. Introduction. Several surprises are in store for the mathematics student who looks for the first time at nontrivial constrained optimization problems in economics. The usual constrained problem in a mathematics course has only one or two critical points and the selection of the absolute maximum is clear from the geometric nature of the problem. Mathematics texts often ignore sufficient conditions (involving bordered Hessian determinants) for relative extrema and provide no interpretation of the Lagrange multiplier λ, leaving the student with the impression that λ has no significance beyond providing an extra variable which magically transforms the constrained problem into an unconstrained higher dimensional problem.

Our purpose here is to examine carefully one example in order to highlight the following features that are typical in the formulation of problems in economics:

(i) Functions are not usually explicitly given, but are assumed to have characteristic qualitative properties. Thus the problems have an air of "theory" rather than "computation." Yet the problems are meaningful and often generate important insights into economic behavior.

(ii) Normally the economist is not interested in solving for a constrained optimum; rather his starting assumption is that an optimum is achieved and seeks to base predictions of behavioral responses on the assumption that optimizing will continue. For example, assuming a firm minimizes the cost of producing a given output, one wishes to know how changes in input prices will affect the firm's behavior. Thus the problem is not "find the minimum," but "assuming the minimum is attained, what consequences can be deduced?" An interesting result is that the economist is primarily interested in necessary conditions for an optimum and wishes profoundly that sufficient conditions were really necessary.

(iii) The Lagrange multiplier λ generally has significance in economic problems. It can usually be interpreted as the rate of change of the optimal value relative to some parameter.

(iv) The implicit function theorem plays a pivotal role in the problem analysis, both theoretically and computationally, since the analysis requires solving a system of nonlinear equations for the endogenous (dependent) variables and computing partial derivatives of these variables with respect to the exogenous (independent) variables.

2. An Illustrative Example. We analyze in detail an example in utility maximization. We begin with the problem description.

Assume an individual obtains utility (i.e., satisfaction) from the consumption of two goods x and y which are purchased in the marketplace in quantities X and Y, respectively, and from a certain quantity L of leisure time l. The theory of consumer choice can then be characterized mathematically as a constrained optimization problem in which the individual chooses, in an optimal way, the amount of leisure time l (and implicitly money income) and the quantities of x and y consumed. We shall assume that the individual spends all his income on the purchases of such goods. If this offends readers who are avid savers, they may assume that one of the consumption goods represents savings. To keep the problem manageable, however, we proceed on the assumption of a two-good world. Let P_x and P_y be the price per unit of x and y, respectively. Let T represent the total time available and L the amount of leisure time chosen by the individual per period (e.g., per week, month, or year). Work time can then be defined as $T - L$. If the market-determined wage rate is w, the individual's income is then $w(T - L)$. Since income equals consumption expenditures, the budget constraint becomes:

(1) $w(T - L) - P_x X - P_y Y = 0.$

We assume that the utility U obtained from a particular choice of X, Y, and L is given by a utility function (unique to the individual)

(2) $U = u(X, Y, L).$

Thus the consumer's problem is to choose X, Y, L to maximize U subject to the constraint (1). Furthermore it is reasonable to assume that

(3) $u_X > 0, \qquad u_Y > 0, \qquad u_L > 0,$

(4) $u_{XX} < 0, \qquad u_{YY} < 0, \qquad u_{LL} < 0,$

(5) $u_{XL} > 0, \qquad u_{YL} > 0,$

(6) either $u_{XY} > 0$ or $u_{XY} < 0$ or $u_{XY} = 0,$

where subscripts denote, as usual, partial derivatives. The first order derivatives in (3) are called marginal utilities. The inequalities in (3) state that utility increases with higher levels of consumption of X and Y and with more leisure time per period. The inequalities in (4) state the "law of diminishing marginal utility." That is, while utility increases with consumption of x, y, and l, it increases at a diminishing rate. The third beer one drinks within an hour does not quench one's thirst as much as the second beer. The inequalities in (5) state that the satisfaction one derives from consuming larger amounts of x (or y) is enhanced by the availability of more leisure time. It takes time to "enjoy things." The sign of u_{XY} in (6) depends on whether x and y are substitutes (pepsi and coke), complements (tennis rackets and tennis lessons), or unrelated (mathematics texts and jellybeans). If the goods are substitutes, $u_{XY} < 0$; if complements, $u_{XY} > 0$; and if unrelated, $u_{XY} = 0$. The inequalities in (3) through (6) are posited relationships from economic theory. [See problem feature (i).]

Applying the method of Lagrange, we introduce the multiplier λ and form the Lagrangian

(7) $v(X, Y, L\ \lambda) = u(X, Y, L) + \lambda(w(T - L) - P_x X - P_y Y).$

Assuming the consumer maximizes utility, the optimal quantities X^*, Y^*, L^* and the multiplier λ^* necessarily satisfy the first-order conditions:

$$\begin{aligned}
&\text{(8a)} \quad v_\lambda = w(T - L) - P_x X - P_y Y = 0,\\
&\text{(8b)} \quad v_X = u_X - \lambda P_x \qquad\qquad\qquad = 0,\\
&\text{(8c)} \quad v_Y = u_Y - \lambda P_y \qquad\qquad\qquad = 0,\\
&\text{(8d)} \quad v_L = u_L - \lambda w \qquad\qquad\qquad\; = 0.
\end{aligned}$$

Observe that because of (3) each of the equations (8b–d) imply that the optimal value $\lambda^* > 0$. [See problem feature (ii).]

In the model, the individual chooses X, Y, and L. The prices P_x, P_y, and the wage rate w are given by market conditions beyond the individual's influence or control. T is given by nature. The economist refers to X, Y, and L as endogenous variables and to P_x, P_y, and w as exogenous variables. The central question for the economist is to analyze the impact on the endogenous variables of changes in the exogenous variables, assuming the individual responds to such changes by making a different set of choices in order to re-maximize utility in the changed context. Thus, if P_x should increase, with P_y and w constant, we might intuitively expect the consumer to adjust by decreasing X and increasing Y or decreasing L in order to earn more income to finance purchases of the more expensive x. Mathematically, we want to solve the four equations (8) for X, Y, L, and λ in terms of P_x, P_y, and w and compute the twelve partial derivatives: $\partial X/\partial P_x$, $\partial X/\partial P_y$, $\partial X/\partial w$, etc. These partial derivatives are referred to as the comparative statics of the model. The model's usefulness is determined by how accurately it predicts adjustments in consumer behavior.

The problem at hand consists of four equations (8a–d) in seven variables. Since we shall assume that the left side of each equation is continuously differentiable and that solutions exist, then by the implicit function theorem, X^*, Y^*, L^*, λ^* will each be continuously differentiable functions of P_x, P_y, and w, if the Jacobian matrix

$$\text{(9)} \quad H = \begin{bmatrix} 0 & -P_x & -P_y & -w \\ -P_x & u_{XX} & u_{XY} & u_{XL} \\ -P_y & u_{YX} & u_{YY} & u_{YL} \\ -w & u_{LX} & u_{LY} & u_{LL} \end{bmatrix}$$

is nonsingular at the optimum point (X^*, Y^*, L^*).

For the moment, let us assume that H is nonsingular. Solving (8) for X, Y, L, λ in terms of P_x, P_y, w and substituting the results into (2), we can express the optimized utility U^* in terms of P_x, P_y, w: $U^* = \tilde{u}(P_x, P_y, w)$. Using the chain rule, we obtain

$$\text{(10)} \quad \frac{\partial U^*}{\partial w} = u_X \frac{\partial X}{\partial w} + u_Y \frac{\partial Y}{\partial w} + u_L \frac{\partial L}{\partial w};$$

then from equations (8) and (10), we obtain

$$\text{(11)} \quad \frac{\partial U^*}{\partial w} = \lambda\left[P_x \frac{\partial X}{\partial w} + P_y \frac{\partial Y}{\partial w} + w \frac{\partial L}{\partial w}\right].$$

From 8(a), we have the identity

$$wT = P_x X + P_y Y + wL,$$

and differentiation yields

$$T = P_x \frac{\partial X}{\partial w} + P_y \frac{\partial Y}{\partial w} + w \frac{\partial L}{\partial w} + L,$$

which allows us to rewrite (11) as

$$\text{(12a)} \quad \lambda = \frac{\partial U*}{\partial w} \frac{1}{T - L}.$$

Thus the Lagrange multiplier λ may be interpreted as the marginal utility of the wage rate per hour of work. Letting

$$B = w(T - L)$$

be the money income of the individual, we can transform (12a) into an expression of more interest in economics. Recalling that the optimum choice of L depends on P_x, P_y, w, we may substitute to express B in terms of P_x, P_y, w:

$$B = f(P_x, P_y, w).$$

We wish to solve this last equation for w in terms of P_x, P_y, B and substitute the result to express $U*$ in terms of P_x, P_y, B:

$$U* = \tilde{u}(P_x, P_y, w) = \tilde{\tilde{u}}(P_x, P_y, B).$$

Then to compute $\partial U*/\partial B = \tilde{\tilde{u}}_B(P_x, P_y, B)$, we of course hold P_x, P_y constant and use the one-dimensional chain rule to obtain

$$\frac{\partial U*}{\partial B} = \frac{\partial U*}{\partial w} \frac{\partial w}{\partial B} = \tilde{u}_w(P_x, P_y, w) \frac{\partial w}{\partial B}.$$

Proceeding to the details, we put

$$h(P_x, P_y, w, B) \equiv f(P_x, P_y, w) - B.$$

We may apply the implicit function theorem to solve $h(P_x, P_y, w, B) = 0$ for w if

$$h_w(P_x, P_y, w, B) = f_w(P_x, P_y, w) \neq 0,$$

after which

$$\frac{\partial w}{\partial B} = \frac{-h_B(P_x, P_y, w, B)}{h_w(P_x, P_y, w, B)} = \frac{1}{f_w(P_x, P_y, w)}.$$

Since $f(P_x, P_y, w) = w(T - L)$, it follows that

$$\begin{aligned} f_w(P_x, P_y, w) &= T - L + w \frac{\partial}{\partial w}(T - L) \\ &= (T - L)\left[1 + \frac{w}{T - L}\frac{\partial}{\partial w}(T - L)\right] \\ &= (T - L)(1 + \varepsilon_S), \end{aligned}$$

where

$$\varepsilon_S \equiv \frac{w}{T - L}\frac{\partial}{\partial w}(T - L)$$

is called the individual's elasticity of supply of labor with respect to wage rate; ε_S may be interpreted as the ratio of a percentage change in work time to a percentage change in wage rate. Thus if $\varepsilon_S \neq -1$, then $f_w(P_x, P_y, w) \neq 0$,and we have

$$\frac{\partial U^*}{\partial B} = \frac{1}{(T - L)(1 + \varepsilon_S)}\frac{\partial U^*}{\partial w},$$

from which

$$\text{(12b)} \quad \lambda = (1 + \varepsilon_S)\,\frac{\partial U^*}{\partial B}.$$

Thus, in this model, λ is proportional to the marginal utility of income. The proportionality constant is one plus the elasticity of labor supply. Note that if there is no supply response to a change in the wage rate, i.e., $\varepsilon_S = 0$, the Lagrange multiplier equals the marginal utility of income. We shall return to this finding later in the paper. [See problem feature (iii).]

Returning to the Jacobian matrix (9), since P_x, P_y, w are all positive, we see from (4) and (5) that regardless of what assumption is made about (6), one cannot guarantee that (9) is nonsingular. If one does the tedious work of expanding and analyzing the determinant of (9), one finds that some terms are positive and some terms are negative. However, widespread economic folklore has held for some time that the negative terms will in practice dominate and that the determinant of H may be safely assumed negative. We shall return to a critique of this assumption in the next section.

We are now at the point where the economist deeply wishes that the sufficient conditions be necessary. The Jacobian matrix in (9) is actually a bordered Hessian. According to the mathematical theory of constrained optimization problems, the analogue of the second derivative test is: The sufficient conditions for a solution X^*, Y^*, L^*, λ^*, of (8) to yield a relative maximum are (a) the determinant of (9) be negative and (b) the determinant of the 3×3 Hessian matrix

$$\begin{bmatrix} 0 & -P_x & -P_y \\ -P_x & u_{XX} & u_{XY} \\ -P_y & u_{YX} & u_{YY} \end{bmatrix}$$

be positive at the point X^*, Y^*, L^*, λ^*.

Assuming that these sufficient conditions actually occur, then not only is the determinant of (9) nonzero, so that the implicit function theorem can be invoked legitimately, but further results of economic interest can be derived. To illustrate, we shall derive several of the twelve comparative statics mentioned above using the implicit function theorem. [See problem feature (iv).]

Let F be the vector-valued function defined for points

$$(\lambda, X, Y, L, P_x, P_y, w) \in \boldsymbol{R}^7$$

and taking values in $\boldsymbol{R}^4$ whose components are given by the left sides of equations (8). By the implicit function theorem, the equation

$$\text{(8)} \quad F(\lambda, X, Y, L, P_x, P_y, w) = 0$$

may be solved in the form

$$\text{(13)} \quad \begin{bmatrix} \lambda \\ X \\ Y \\ L \end{bmatrix} = G(P_x, P_y, w).$$

Furthermore, the Jacobian matrix for G is given by

$$\begin{bmatrix} \dfrac{\partial \lambda}{\partial P_x} & \dfrac{\partial \lambda}{\partial P_y} & \dfrac{\partial \lambda}{\partial w} \\[2ex] \dfrac{\partial X}{\partial P_x} & \dfrac{\partial X}{\partial P_y} & \dfrac{\partial X}{\partial w} \\[2ex] \dfrac{\partial Y}{\partial P_x} & \dfrac{\partial Y}{\partial P_y} & \dfrac{\partial Y}{\partial w} \\[2ex] \dfrac{\partial L}{\partial P_x} & \dfrac{\partial L}{\partial P_y} & \dfrac{\partial L}{\partial w} \end{bmatrix} = -H^{-1} \begin{bmatrix} -X & -Y & T-L \\ -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix},$$

where the ith row in the last matrix on the right is obtained by differentiating the ith left side in (8) with respect to P_x, then P_y, and then w. Letting C_{ij} be the cofactor (signed minor determinant) of the element in the ith row and jth column of H, and then inverting H using the method of cofactors gives

$$H^{-1} = \frac{1}{\det H} C^T,$$

where $C = (C_{ij})$. Now we are in a position to derive comparative static results. For example, how does the level of consumption of X change when its price increases? We find that

$$\text{(14)} \quad \frac{\partial X}{\partial P_x} = \frac{-1}{\det H} [-XC_{12} - \lambda C_{22}].$$

Computing C_{12} and C_{22}, we obtain,

$$C_{22} = -P_y^2 u_{LL} - w^2 u_{YY} + 2wP_y u_{LY}$$

and

$$C_{12} = P_x[u_{YY} u_{LL} - (u_{Ly})^2] \\ - P_y[u_{XY} u_{LL} - u_{LY} u_{XL}] + w[u_{XY} u_{YL} - u_{YY} u_{XL}].$$

Because of (4) and (5), $C_{22} > 0$. Since $u_{XY} \geq 0$ (x and y are complements or unrelated), the second and third terms of C_{12} are unambiguously positive. If we assume also that

$$\det \begin{bmatrix} u_{LL} & u_{LY} \\ u_{YL} & u_{YY} \end{bmatrix} = u_{LL} u_{YY} - (u_{LY})^2 \geq 0,$$

as economists generally do (we examine this assumption in Section 3), then $C_{12} > 0$. Thus $\partial X/\partial P_x < 0$, which is intuitively correct and is certainly in accord with empirical evidence.

Next, we might ask: How does the amount of leisure time chosen change when the wage rate increases? We find that

$$(15) \quad \frac{\partial L}{\partial w} = \frac{-1}{\det H}[-\lambda C_{44} + (T - L)C_{14}],$$

where

$$C_{44} = -(P_x^2 u_{XX} + P_y^2 u_{YY}) + 2P_x P_y u_{XY}$$

and

$$C_{14} = P_x(u_{YX} u_{LY} - u_{LX} u_{YY}) - P_y(u_{XX} u_{LY} - u_{LX} u_{XY}) \\ + w(u_{XX} u_{YY} - (u_{XY})^2).$$

Continuing to assume that $u_{XY} \geq 0$, we infer from assumptions (4) and (5) that C_{44} is unambiguously positive. Furthermore, the first and second terms in $C_{14}$are positive. Assuming again with the economists that

$$\det \begin{bmatrix} u_{XX} & u_{XY} \\ u_{YX} & u_{YY} \end{bmatrix} = u_{XX} u_{YY} - (u_{XY})^2 \geq 0,$$

we see that C_{14} is positive. Therefore the sign of $\partial L/\partial w$ is ambiguous. However, this is itself a very interesting result.

Economists refer to the first term in the brackets on the right side of (15), $-\lambda C_{44}$, as the *substitution effect* of a wage increase. When wages increase there is a stronger incentive (reward) to work longer hours, i.e., to choose less leisure time. The second term in the brackets on the right side of (15), $(T - L)C_{14}$, is called the *income effect* by which is meant that a wage hike increases income and allows an individual to opt for more leisure in order to have the time to enjoy the goods he consumes. For example, a 10% wage hike might lead an individual to choose 5% fewer work hours plus a 5% higher income. In general a wage hike gives rise to both income and substitution effects which have opposite influences on an individual choosing a new optimal amount of leisure per period.

However, (15) can be given an even more precise economic interpretation. Using (12a), which defines λ in economic terms, we can rewrite (15),

$$(16)\quad \frac{\partial L}{\partial w} = \frac{-1}{\det H}\frac{1}{(T-L)}\left[-u_w C_{44} + (T-L)^2 C_{14}\right].$$

From (16), we can see qualitatively that, if u_w is small or if an individual works long hours ($T - L$ is large), the income effect dominates and $\partial L/\partial w > 0$. Conversely, if u_w is large *or* if he works few hours ($T - L$ is small), the substitution effect dominates and $\partial L/\partial w < 0$. Put another way, relatively high income, whether because of a high wage rate and/or long hours of work, tends to lead a person to opt for more leisure when w rises.

Finally, with a little more manipulation, we can give a specific interpretation to a zero elasticity of labor supply. Equation (16) can be rewritten

$$\frac{-\partial(T-L)}{\partial w} = \frac{-1}{\det H}\frac{1}{(T-L)}\left[-u_w C_{44} + (T-L)^2 C_{14}\right]$$

or

$$(17)\quad \varepsilon_S = \frac{w}{(T-L)}\frac{\partial(T-L)}{\partial w}$$

$$= \frac{1}{\det H}\frac{w}{(T-L)^2}\left[-u_w C_{44} + (T-L)^2 C_{14}\right].$$

But (17) states that when $T \neq L$, $\varepsilon_S = 0$ if and only if the income effect

$$\left[(T-L)^2 C_{14}\right]$$

just equals the substitution effect $\left[-u_w C_{44}\right]$ of a wage change.

These comparative static results for $\partial X/\partial P_x$ and $\partial L/\partial w$ are intuitively appealing and they are important. For example, one implication of this analysis is that an income tax cut designed to stimulate work effort, by raising after tax wage rates ("supply side economics"), may generate little response because the tax cut will give rise to both substitution *and* income effects.

3. Discussion of Assumptions. The main reason for the assumptions made by applied mathematicians is that they allow a conclusion to be drawn. In practice, the ends justify any (reasonable) means. The real reason that physicists neglect friction, air resistance, etc., is to make nonlinear problems linear or very difficult problems solvable in closed form. Of course, any scientist with integrity will then examine the validity of his assumptions.

Although the differential equations textbooks of the past twenty-five years are significantly superior to their predecessors, one of these older books by Agnew [**1**] is relevant to the present discussion and, if the reader can locate a copy, the incisive humorous comments there about assumptions in applied mathematics provide insight. Agnew's famous discussion of the "snowplow problem" (Section 2.8) contains the opinion "much of the progress in science is due to men who have the courage to make assumptions, the good sense to make reasonable assumptions, and the ability to draw correct conclusions from the assumptions." In Section 3.4 on atmospheric pressures, he says "we make progress by application of a time honored method which often brings valuable results in applied mathematics. This method consists in making an assumption known to be absolutely and unequivocally false." But we digress.

We first examine the determinant of H (see (9)) in detail. Expanding the determinant by minors across the first row, then expanding each of the resulting 3×3 determinants down the first column, and finally simplifying, we get

$$(18) \quad \det H = -P_x^2 \det \begin{bmatrix} u_{YY} & u_{YL} \\ u_{LY} & u_{LL} \end{bmatrix} - P_y^2 \det \begin{bmatrix} u_{XX} & u_{XL} \\ u_{LX} & u_{LL} \end{bmatrix}$$
$$- w^2 \det \begin{bmatrix} u_{XX} & u_{XY} \\ u_{YX} & u_{YY} \end{bmatrix} + 2P_x P_y \det \begin{bmatrix} u_{XY} & u_{XL} \\ u_{LY} & u_{LL} \end{bmatrix}$$
$$- 2P_x w \det \begin{bmatrix} u_{XY} & u_{XL} \\ u_{YY} & u_{YL} \end{bmatrix} + 2P_y w \det \begin{bmatrix} u_{XX} & u_{XY} \\ u_{LX} & u_{LY} \end{bmatrix}.$$

Observe that the first three determinants on the right side of (18) are of similar type and two of them were assumed non-negative in our discussions of (14) and (15).

Let us call two goods x and y weakly dependent (with respect to the utility function u) if

$$\det \begin{bmatrix} u_{XX} & u_{XY} \\ u_{YX} & u_{YY} \end{bmatrix} \geq 0.$$

As always, X (or Y) denotes a quantity of good x (or y). If we assume that each pair of the three goods, x, y, l are weakly dependent and x and y are complements or unrelated ($u_{XY} \geq 0$), it is an immediate consequence of (4) and (5) that $\det H < 0$. However, if $u_{XY} < 0$, then (18) contains positive terms which threaten to change the sign of $\det H$, or make $\det H = 0$. However, we emphasize that our comparative static results in Section 2 are firmly established if each pair of x, y, l are weakly dependent and x, y are complements or unrelated.

Let us examine the concept of weak dependence more closely. Widespread economic folklore has held that for realistic economic situations, the inequalities

$$(19) \quad |u_{XY}| \leq |u_{XX}|, \; |u_{XY}| \leq |u_{YY}|$$

would surely hold. One rationalization of such inequalities runs as follows. "Consider the extreme cases. If x and y are unrelated, $u_{XY} = 0$ and the inequalities are trivial. If x and y are perfect substitutes (for all practical purposes, indistinguishable), then $u_{XY} = u_{XX} = u_{YY}$ and the inequalities are equations. Other cases lie between these two extremes."

Of course, this argument does not stand up under close examination. In fact, the inequalities in (19) are not generally true and are not really required. The required inequality is

$$(20) \quad \det \begin{bmatrix} u_{XX} & u_{XY} \\ u_{YX} & u_{YY} \end{bmatrix} = u_{XX}u_{YY} - (u_{XY})^2 \geq 0.$$

Note that (19) implies (20) but the converse is not true.

To illustrate the difficulty with (19), suppose that $u_{XY} = 0.1$, $u_{XX} = -1.0$ at some fixed point. If we change Y units to a new unit Y' where $Y = 12Y'$ (for example, Y units are inches, Y' units are feet), then

$$u_{XY'} = u_{XY}\frac{dY}{dY'} = 12u_{XY} = 1.2$$

and thus $|u_{XY'}| \leq |u_{XX}|$ fails using the new units.

It is easy to verify that the sign of a determinant like the one in (20) or det H in (9) is not affected by changing units of measurement. For example, if X units were yards and $X' = 3X$ measured in feet, the effect of changing X units to X' units on (9) would be to multiply the second column and the second row of H by 3, thus changing det H by a factor of $1/9$, leaving the sign of det H unchanged.

If x, y are substitutes $(u_{XY} < 0)$, but u_{XY} is sufficiently close to 0, det $H < 0$ will continue to hold. Comparative static conclusions change in this case, as the reader may verify.

4. Other Representative Problems from Economics. The approach we have just illustrated is useful in addressing a variety of problems in economics. We suggest several other types of problems below, which may be explored by the reader or students in independent study projects. One would begin by applying the Lagrange method to see which features of the previous problem reappear, perhaps with modification.

(a) *Cost Minimization.* Suppose a firm is under contract to produce and deliver (for a fixed price) Q_0 units of output during the next year. It employs capital (K), labor (L), and other inputs (R) in its production process. If the firm seeks to maximize profits while meeting the terms of the contract, its production decision can be characterized as a constrained cost minimization problem in which the firm chooses the least cost combination of K, L, and R necessary to produce Q_0.

Its objective is then to minimize the cost function

$$C(K, L, R) = rK + wL + \rho R$$

subject to the output constraint

$$Q_0 = f(K, L, R)$$

when r, w, and ρ are the cost per unit of K, L, and R, respectively, and f is the production function for the firm. Some obvious assumptions need to be formulated regarding the qualitative behavior of f, similar to those we made earlier on the utility function. In this problem, the Lagrange multiplier turns out to be the marginal cost of production $\partial C/\partial Q_0$, that is, the increase in total costs when one more unit of output is produced.

(b) *Output Maximization.* Suppose a government agency is given an annual budget B_0 and is charged with the responsibility of providing as much service to the community as possible. If the agency uses K, L, and R to produce services, then its objective is to maximize the output function

$$Q = g(K, L, R)$$

subject to the budget constraint

$$B_0 = rK + wL + \rho R.$$

(The meaning of the various parameters should be clear.) In this problem, the Lagrange multiplier is $\partial Q/\partial B_0$, the inverse of marginal cost. This problem is thus dual to the cost minimization problem.

(c) *Multiple Constraints.* During wartime, consumers often must pay for goods with money and the surrender of ration coupons. The mathematical problem is then (using an obvious notation) to maximize an individual's utility function $U(x, y, z)$ subject to the budget constraint

$$B = P_x X + P_y Y + P_z Z$$

and the coupon constraint

$$R = r_x X + r_y Y + r_z Z.$$

(d) *Nonlinear Constraints.* Suppose our government agency in (b) can obtain price discounts by purchasing larger quantities of R. We can rewrite the budget constraint as

$$B_0 - rK - wL - \rho(R)R = 0.$$

5. Concluding Comments. We provide several references which the interested reader may wish to consult. Statements of sufficient conditions for local maxima (or minima) can be found in [**3**] and [**6**]. A very readable account of some elementary but interesting applications in economics can be found in [**4**]. An interpretation of the Lagrange multiplier and some consequences of the necessary conditions are provided in [**7**] but no use of sufficient conditions is mentioned there.

The authors of this paper have for several years jointly taught an upper level seminar in mathematical economics as part of an interdisciplinary major. The type of problem described here is considered in the early weeks of the seminar. Because each of us has learned a lot of useful things from the other, we recommend such interdisciplinary endeavors with enthusiasm.

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