Proof of Fact 1. Looking at right triangle BDE, we have $DE = (BD)\tan\alpha$, where $\alpha = \angle DBE$, or

$$x = (1-y)\tan\alpha. \tag{3}$$

From triangle ACB and the law of sines, we have

$$\sin(\theta+\alpha) = \frac{y+2r}{2r}\sin\theta, \tag{4}$$

so that $\alpha = \sin^{-1}\left(\left(1+\frac{y}{2r}\right)\sin\theta - \theta\right)$. This gives

$$x = (1-y)\tan\left(\sin^{-1}\left(\left(1+\frac{y}{2r}\right)\sin\theta - \theta\right)\right).$$

Proof of Fact 2. For this we need just the usual approximations:

$$u \approx \sin u \approx \tan u \approx \sin^{-1} u, \text{ for } u \to 0.$$

Proof of Fact 3. $\left(\frac{(1-y)y}{2r}\theta\right)$ is quadratic in y with its maximum at the average of its zeros, namely, $y = 1/2$.

Proof of Fact 4. Let α_k denote the angle of deflection of the k'th ball after it is contacted by the $(k-1)$'st, where the cue ball is the 0'th ball. The following are easy to verify:

$$\begin{aligned}
\frac{x}{1-y_n} &= \tan\alpha_n & \alpha_n &\approx \frac{x}{1-y_n} \\
\sin(\alpha_n+\alpha_{n-1}) &= (y_n-y_{n-1})\frac{\sin\alpha_{n-1}}{2r} & \alpha_n+\alpha_{n-1} &\approx \frac{y_n-y_{n-1}}{2r}\alpha_{n-1} \\
&\vdots & &\vdots \\
\sin(\alpha_k+\alpha_{k-1}) &= (y_k-y_{k-1})\frac{\sin\alpha_{k-1}}{2r} & \alpha_k+\alpha_{k-1} &\approx \frac{y_k-y_{k-1}}{2r}\alpha_{k-1} \\
&\vdots & &\vdots \\
\sin(\alpha_2+\alpha_1) &= (y_2-y_1)\frac{\sin\alpha_1}{2r} & \alpha_2+\alpha_1 &\approx \frac{y_2-y_1}{2r}\alpha_1 \\
\sin(\alpha_1+\theta) &= (y_1+2r)\frac{\sin\theta}{2r} & \alpha_1+\theta &\approx \frac{y_1+2r}{2r}\theta.
\end{aligned}$$

Solving gives $x \approx (1-y_n)\left(\frac{y_n-y_{n-1}-2r}{2r}\right)\cdots\left(\frac{y_2-y_1-2r}{2r}\right)\left(\frac{y_1}{2r}\right)\theta$. Letting $z_{n+1} = 1-y_n$, $z_k = y_k-y_{k-1}-2r$ (for $1<k<n$) and $z_1 = y_1$, we want to maximize $z_1z_2\cdots z_{n+1}$ subject to $z_1+z_2+\cdots+z_{n+1} = 1-2r$. This is standard (by Lagrange multipliers, say) resulting in

$$z_1 = z_2 = \cdots = z_{n+1} = \frac{1-2r}{n+1}.$$

Proof of Fact 5. Holding θ and r constant, let $f(y) = x(y, \theta, r)$ and differentiate (3) and (4) with respect to y to get

$$f'(y) = -\tan\alpha + (1-y)\sec^2\alpha \frac{d\alpha}{dy}$$

and

$$\frac{d\alpha}{dy} = \frac{\sin\theta}{2r}\sec(\theta+\alpha).$$

Differentiating each once more gives

$$f''(y) = -2\sec^2\alpha \frac{d\alpha}{dy} + (1-y)\sec^2\alpha \left(2\tan\alpha \left(\frac{d\alpha}{dy}\right)^2 + \frac{d^2\alpha}{dy^2}\right)$$

and

$$\frac{d^2\alpha}{dy^2} = \frac{\sin\theta}{2r}\sec(\alpha+\theta)\tan(\alpha+\theta)\frac{d\alpha}{dy}.$$

Combining these gives

$$\frac{d^2x}{dy^2} = \frac{d\alpha}{dy}(\sec^2\alpha)(-2+\mathcal{J}),$$

where

$$\mathcal{J} = (1-y)\frac{\sin\theta}{2r}\sec(\alpha+\theta)(2\tan\alpha + \tan(\alpha+\theta)).$$

We'll show that $\mathcal{J} < 3/2$, which will prove that $f''(y) < 0$.
Since $\tan\alpha < \tan(\alpha+\theta)$, we have

$$\begin{aligned}
\sec(\alpha+\theta)(2\tan\alpha+\tan(\alpha+\theta)) &< 3\sec(\alpha+\theta)\tan(\alpha+\theta) \\
&= \frac{3\sin(\alpha+\theta)}{1-\sin^2(\alpha+\theta)} \\
&= \frac{3(\sin\theta)(1+\frac{y}{2r})}{1-(\sin^2\theta)(1+\frac{y}{2r})^2}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\mathcal{J} &< \frac{(1-y)}{2r}\,\frac{3\left(\frac{2r}{1+2r}\right)^2\left(1+\frac{y}{2r}\right)}{1-\left(\frac{2r}{1+2r}\right)^2\left(1+\frac{y}{2r}\right)^2} \\
&= \frac{3(2r+y)}{1+4r+y} \\
&< \frac{3}{2}.
\end{aligned}$$

Proof of Fact 6. $f(y)$ has a unique maximum for $0 \leq y \leq 1$, since $f(0) = f(1) = 0$ and $f''(y) < 0$ on this interval. By showing $f'(1/2) > 0$, it will follow that this maximum occurs for $1/2 < y < 1$.

We've already noted (implictly) the dependence of α on y, but let's set $y = 1/2$ in (4) to get

$$\sin(\alpha + \theta) = (\sin\theta)\left(1 + \frac{1}{4r}\right),$$

and continue to write α for the specific value of α so obtained (which still depends on the fixed values of θ and r). Using the fact that $\sec^2\alpha > \sec\alpha$, our formula for $f'(y)$ from a previous napkin gives

$$\begin{aligned} f'(1/2) &> -\tan\alpha + \frac{1}{4r}\sec\alpha\sin\theta\sec(\alpha+\theta) \\ &= \sec\alpha(-\sin\alpha + (\sin(\alpha+\theta) - \sin\theta)\sec(\alpha+\theta)) \\ &= \sec\alpha\sec(\alpha+\theta)\cdot\mathcal{L}, \end{aligned}$$

where $\mathcal{L} = \sin(\alpha+\theta) - \sin\alpha\cos(\alpha+\theta) - \sin\theta$. We're done if $\mathcal{L} > 0$.

We note that $\mathcal{L} = 0$ for $\alpha = 0$, so we'll be done if $\dfrac{\partial\mathcal{L}}{\partial\alpha} > 0$. And indeed,

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\alpha} &= \cos(\alpha+\theta) - \cos\alpha\cos(\alpha+\theta) + \sin\alpha\sin(\alpha+\theta) \\ &> \cos(\alpha+\theta) - \cos(\alpha+\theta) + \sin\alpha\sin(\alpha+\theta) \\ &= \sin\alpha\sin(\alpha+\theta) \\ &> 0. \end{aligned}$$

Proof of Fact 7. We want to show that $f'(0) < -f'(1)$. Letting $a = 1/(2r)$, this translates into proving that $a \tan\theta < \tan((\sin^{-1}((a+1)\sin\theta) - \theta)$, or

$$\tan^{-1}(a\tan\theta) + \theta < \sin^{-1}((a+1)\sin\theta). \tag{5}$$

Note that $a > 1$ and that $(a+1)\sin\theta > 1$ (our condition of the maximum angle). Both sides of the inequality in (5) are zero for $\theta = 0$, so we are finished if the inequality holds when differentiated. That is, we are done if we can show that

$$1 + \frac{a\sec^2\theta}{1 + a^2\tan^2\theta} < \frac{(a+1)\cos\theta}{\sqrt{1-(a+1)^2\sin^2\theta}}. \tag{6}$$

Squaring both sides, cross-multiplying, then gathering everything to the right (brute-force here; I won't say if I had any electronic assistance), our inequality in (6) is true if

$$a^2\sin^2\theta(3 - (a^2+2a+3)\sin^2\theta) > 0.$$

Again using the fact that $\sin\theta < \frac{1}{a+1}$, we have

$$\begin{aligned} 3 - (a^2+2a+3)\sin^2\theta &> 3 - \frac{a^2+2a+3}{(a+1)^2} \\ &= \frac{2a(a+2)}{(a+1)^2}, \end{aligned}$$

which is positive, so we're done.

Proof of Fact 8. Since $f(1-y) > f(y) \Leftrightarrow \frac{f(1-y)}{y(1-y)} > \frac{f(y)}{y(1-y)}$, by defining $g(y) = \frac{f(y)}{y(1-y)}$, it suffices then to show that g is increasing on $[0, 1/2]$. Using $\sin(\theta+\alpha) = \left(1 + \frac{y}{2r}\right)\sin\theta$, or $\sin(\theta+\alpha) - \sin\theta = \frac{y}{2r}\sin\theta$, and $\sin\theta \le \frac{2r}{y+2r}$, we have,

$$\begin{aligned}
g'(y) &= \frac{f'(y)}{y(1-y)} + \frac{f(y)(2y-1)}{y^2(1-y)^2} \\
&= \frac{-y\tan\alpha + y(1-y)(\sec^2\alpha)\frac{\sin\theta}{r}\sec(\theta+\alpha) + (2y-1)\tan\alpha}{y^2(1-y)} \\
&= \frac{y\sec^2\alpha\frac{\sin\theta}{r}\sec(\theta+\alpha) - \tan\alpha}{y^2} \\
&= \frac{\sec^2\alpha(\sin(\theta+\alpha) - \sin\theta)\sec(\theta+\alpha) - \tan\alpha}{y^2} \\
&> \frac{(\sin(\theta+\alpha) - \sin\theta)\sec(\theta+\alpha) - \tan\alpha}{y^2} \\
&= \frac{\sec\alpha\sec(\theta+\alpha)}{y^2}((\sin(\theta+\alpha) - \sin\theta)\cos\alpha - \cos(\theta+\alpha)\sin\alpha) \\
&= \frac{\sec\alpha\sec(\theta+\alpha)}{y^2}\sin\theta(1-\cos\alpha) \\
&> 0.
\end{aligned}$$

Proof of Fact 9. For each fixed $r \in (0, 1/2)$ and $y \in (0, 1)$, the maximum value of x is

$$x(y, \sin^{-1}\frac{2r}{1+2r}, r) = (1-y)\tan\left(\sin^{-1}\left(\frac{y+2r}{1+2r}\right) - \sin^{-1}\left(\frac{2r}{1+2r}\right)\right).$$

Letting $r \to 0$ in the above gives

$$(1-y)\frac{y}{\sqrt{1-y^2}},$$

a quantity which is zero when $y = 0$ and for $y \to 1$, and which is otherwise positive. The derivative of this quantity is

$$\frac{-(1-y)(y^2+y-1)}{(1-y^2)^{3/2}},$$

which has as its single zero in $(0, 1)$ the number we desire.

Our work here is done. Shoot.