Sample *Mathematica* code for the paper "Modeling a Diving Board"

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■ Beam Data

■ Original Data

Remove headings from original data collected with World in Motion software.

NOTE: At t = 1 second, World in Motion recorded values of (-2000, -2000) for the (x, y) data point at the 13th tape mark, at approximately x = 1.46 m. Looking at the nearby data, these clearly cannot be correct, so we interpolated the values from the corresponding points for x = 1.46 m at t = 0.967 sec, which was recorded as (1.472478, 0), and at t = 1.033 sec, which was recorded as (1.481429, 0.0582). These corrections are noted below in red.

```
In[1]:= Beam = Drop[{{"t(s)", "x1(m)", "y1(m)", "x2(m)", "y2(m)", "x3(m)", "y3(m)",
        "x4(m)", "y4(m)", "x5(m)", "y5(m)", "x6(m)", "y6(m)", "x7(m)", "y7(m)",
        "x8(m)", "y8(m)", "x9(m)", "y9(m)", "x10(m)", "y10(m)", "x11(m)", "y11(m)",
        "x12(m)", "y12(m)", "x13(m)", "y13(m)", "x14(m)", "y14(m)", "x15(m)", "y15(m)"},
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```

```
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```

Remove the first entry of each row, which corresponds to the time t at which the data was collected, for t ≥ 0 seconds.

```
In[2]:= Beam = Drop[Beam, {}, {1}];
```

Define the positions for the tape pieces along the beam. Check the length of this list - there should be fifteen points.

```
In[3]:= xtape = Table[Beam[[1]][[i]], {i, 1, Length[Beam[[1]]], 2}]

Out[3]= {0, 0.107415, 0.219305, 0.340147, 0.447562, 0.554977, 0.671343,
 0.783233, 0.899599, 1.01597, 1.12338, 1.2308, 1.34716, 1.45905, 1.57094}

In[4]:= Length[xtape]

Out[4]= 15
```

■ Steady-State Solution

Solve the steady-state problem.

```
In[5]:= Clear[L, v, c, g]

In[6]:= DSolve[{-c^2 V''''[x] == g, V[0] == 0, V'[0] == 0, V''[L] == 0, V'''[L] == 0}, V[x], x]
```

Out[6]= $\left\{\left\{V[x] \to \frac{-6 g L^2 x^2 + 4 g L x^3 - g x^4}{24 c^2}\right\}\right\}$

In[7]:= $v[x\_] := \frac{-6 g L^2 x^2 + 4 g L x^3 - g x^4}{24 c^2}$

```
In[8]:= v[x]
```

Out[8]= $\frac{-6 g L^2 x^2 + 4 g L x^3 - g x^4}{24 c^2}$

Initial Conditions

Define initial conditions and model parameters. Length of beam is L meters. Parameters c, k, and g will be chosen dynamically below. The function f(x) is initial displacement of beam. Initial velocity g(x) is assumed to be zero. Define $h(x) = f(x) - v(x)$ to be the extra amount by which the beam is bent initially.

In[9]:=
```
L = 1.6;
```

In[10]:=
```
Fit[Partition[Beam[[1]], 2], {x, x^2}, x]
```

Out[10]= $-0.0341084\,x - 0.080823\,x^2$

In[11]:=
```
f[x_] := a x - b x^2
```

In[12]:=
```
a = -0.0341084;
b = 0.0808230;
```

In[14]:=
```
f[x]
```

Out[14]= $-0.0341084\,x - 0.080823\,x^2$

In[15]:=
```
h[x_] := f[x] - v[x]
```

In[16]:=
```
h[x]
```

Out[16]= $-0.0341084\,x - 0.080823\,x^2 - \frac{-15.36\,g\,x^2 + 6.4\,g\,x^3 - g\,x^4}{24\,c^2}$

Construct Model

Solve for the alpha values

In[17]:=
```
Clear[alpha, α]
alpha = Table[α /. FindRoot[Cos[α * L] + 1/Cosh[α * L], {α, (2 n + 1) π/(2 L)}], {n, 0, 7}];
α = alpha;
```

Define position functions $X_n(x, t)$

In[20]:=
```
X[x_, n_] := Cos[α[[n]] * x] - Cosh[α[[n]] * x] -
  (Cosh[α[[n]] * L] + Cos[α[[n]] * L]) / (Sinh[α[[n]] * L] + Sin[α[[n]] * L]) (Sin[α[[n]] * x] - Sinh[α[[n]] * x])
```

- **Define the A_n Coefficients - for the Manipulate command to work, the coefficients A_n have been found explicitly, which can be done with our choice of f(x)!**

In[21]:=
```
Clear[A, AN];
A[n_] := ((Sin[L α[[n]]] + Sinh[L α[[n]]]) (-8 g Cos[L α[[n]]] + 8 c^2 L (a - b L) α[[n]]^4 + g (8 + L^4 α[[n]]^4) +
      Cosh[L α[[n]]] (-8 g + Cos[L α[[n]]] (8 c^2 L (a - b L) α[[n]]^4 + g (8 + L^4 α[[n]]^4))) -
      8 c^2 α[[n]]^2 (a α[[n]] Sin[L α[[n]]] + (a α[[n]] - 2 b Sin[L α[[n]]]) Sinh[L α[[n]]]))) /
   (2 c^2 α[[n]]^4 (-L Cos[2 L α[[n]]] α[[n]] + L Cosh[2 L α[[n]]] α[[n]] - 6 Cosh[L α[[n]]] Sin[L α[[n]]] -
      3 Cosh[L α[[n]]]^2 Sin[2 L α[[n]]] + (6 Cos[L α[[n]]] + 4 L α[[n]] Sin[L α[[n]]]) Sinh[L α[[n]]] +
      3 Cos[L α[[n]]]^2 Sinh[2 L α[[n]]]))
AN = Table[A[n], {n, 1, 7}];
```

- **Define the B_n Coefficients**

In[24]:=
```
Clear[μ, B, BN];
μ[n_] := 1/2 Sqrt[4 c^2 α[[n]]^4 - k^2];
B[n_] := (k AN[[n]])/(2 μ[n]);
BN = Table[B[n], {n, 1, 7}];
```

- **Define time functions $T_n(x, t)$**

In[28]:=
```
T[t_, n_] := (AN[[n]] Cos[μ[n] t] + BN[[n]] Sin[μ[n] t]) e^(-1/2 k t);
```

- **Define model $y(x, t) = v(x) + \sum X_n(x, t)\, T_n(x, t)$**

In[29]:=
```
y[x_, t_] := v[x] + Sum[X[x, n] T[t, n], {n, 1, 7}]
```

Check to make sure our model is a function of c, k, and g:

In[30]:=
```
y[x, t]
```

Out[30]=

$$\frac{-15.36\, g\, x^2 + 6.4\, g\, x^3 - g\, x^4}{24\, c^2} + e^{-\frac{k t}{2}} \left(\frac{1}{c^2} 0.01708\, (3.26694\, c^2 + 22.7594\, g + 3.33742\, (-8\, g - 0.299633\, (-3.94594\, c^2 + 20.3624\, g))) \operatorname{Cos}\left[\frac{1}{2}\sqrt{7.54539\, c^2 - k^2}\; t\right] + \frac{1}{c^2 \sqrt{7.54539\, c^2 - k^2}} 0.01708\, (3.26694\, c^2 + 22.7594\, g + 3.33742\, (-8\, g - 0.299633\, (-3.94594\, c^2 + 20.3624\, g)))\; k \operatorname{Sin}\left[\frac{1}{2}\sqrt{7.54539\, c^2 - k^2}\; t\right] \right)$$

$$(\operatorname{Cos}[1.17194\, x] - \operatorname{Cosh}[1.17194\, x] - 0.734096\, (\operatorname{Sin}[1.17194\, x] - \operatorname{Sinh}[1.17194\, x])) + e^{-\frac{k t}{2}}$$

$$\left(\frac{1}{c^2} 0.0000134008\, (-393.464\, c^2 + 493.665\, g + 54.6543\, (-8\, g - 0.0182968\, (-154.973\, c^2 + 493.519\, g))) \right.$$

$$\text{Cos}\left[\frac{1}{2}\sqrt{296.337\,c^2-k^2}\;t\right]+\frac{1}{c^2\sqrt{296.337\,c^2-k^2}}$$

$$0.0000134008\left(-393.464\,c^2+493.665\,g+54.6543\left(-8\,g-0.0182968\left(-154.973\,c^2+493.519\,g\right)\right)\right)$$

$$\left.k\,\text{Sin}\left[\frac{1}{2}\sqrt{296.337\,c^2-k^2}\;t\right]\right)$$

$$(\text{Cos}[2.93381\,x]-\text{Cosh}[2.93381\,x]-1.01847\,(\text{Sin}[2.93381\,x]-\text{Sinh}[2.93381\,x]))+e^{-\frac{k\,t}{2}}$$

$$\left(\frac{1}{c^2}4.24787\times10^{-8}\left(80\,603.5\,c^2+3814.55\,g+1288.99\left(-8\,g-0.000775804\left(-1215.01\,c^2+3814.55\,g\right)\right)\right)\right.$$

$$\text{Cos}\left[\frac{1}{2}\sqrt{2323.33\,c^2-k^2}\;t\right]+\frac{1}{c^2\sqrt{2323.33\,c^2-k^2}}$$

$$4.24787\times10^{-8}\left(80\,603.5\,c^2+3814.55\,g+1288.99\left(-8\,g-0.000775804\left(-1215.01\,c^2+3814.55\,g\right)\right)\right)$$

$$\left.k\,\text{Sin}\left[\frac{1}{2}\sqrt{2323.33\,c^2-k^2}\;t\right]\right)$$

$$(\text{Cos}[4.90922\,x]-\text{Cosh}[4.90922\,x]-0.999224\,(\text{Sin}[4.90922\,x]-\text{Sinh}[4.90922\,x]))+$$

$$e^{-\frac{k\,t}{2}}\left(\frac{1}{c^2}3.42042\times10^{-10}\right.$$

$$\left(814\,491.\,c^2+14\,625.3\,g+29\,803.9\left(-8\,g-0.0000335527\left(-4665.69\,c^2+14\,625.3\,g\right)\right)\right)$$

$$\text{Cos}\left[\frac{1}{2}\sqrt{8921.68\,c^2-k^2}\;t\right]+\frac{1}{c^2\sqrt{8921.68\,c^2-k^2}}3.42042\times10^{-10}\left(814\,491.\,c^2+14\,625.3\,g+\right.$$

$$\left.\left.29\,803.9\left(-8\,g-0.0000335527\left(-4665.69\,c^2+14\,625.3\,g\right)\right)\right)k\,\text{Sin}\left[\frac{1}{2}\sqrt{8921.68\,c^2-k^2}\;t\right]\right)$$

$$(\text{Cos}[6.87221\,x]-\text{Cosh}[6.87221\,x]-1.00003\,(\text{Sin}[6.87221\,x]-\text{Sinh}[6.87221\,x]))+$$

$$e^{-\frac{k\,t}{2}}\left(\frac{1}{c^2}4.20671\times10^{-12}\right.$$

$$\left(1.99439\times10^{8}\,c^2+39\,951.8\,g+689\,706.\left(-8\,g-1.44989\times10^{-6}\left(-12\,749.7\,c^2+39\,951.8\,g\right)\right)\right)$$

$$\text{Cos}\left[\frac{1}{2}\sqrt{24\,379.8\,c^2-k^2}\;t\right]+\frac{1}{c^2\sqrt{24\,379.8\,c^2-k^2}}4.20671\times10^{-12}\left(1.99439\times10^{8}\,c^2+39\,951.8\,g+\right.$$

$$\left.\left.689\,706.\left(-8\,g-1.44989\times10^{-6}\left(-12\,749.7\,c^2+39\,951.8\,g\right)\right)\right)k\,\text{Sin}\left[\frac{1}{2}\sqrt{24\,379.8\,c^2-k^2}\;t\right]\right)$$

$$(\text{Cos}[8.83573\,x]-\text{Cosh}[8.83573\,x]-0.999999\,(\text{Sin}[8.83573\,x]-\text{Sinh}[8.83573\,x]))+$$

$$e^{-\frac{k\,t}{2}}\left(\frac{1}{c^2}6.66526\times10^{-14}\right.$$

$$\left(3.07785\times10^{9}\,c^2+89\,143.4\,g+1.59603\times10^{7}\left(-8\,g-6.26556\times10^{-8}\left(-28\,451.1\,c^2+89\,143.4\,g\right)\right)\right)$$

$$\mathrm{Cos}\left[\frac{1}{2}\sqrt{54\,403.9\,c^2-k^2}\ t\right]+\frac{1}{c^2\sqrt{54\,403.9\,c^2-k^2}}6.66526\times 10^{-14}\left(3.07785\times 10^{9}\,c^2+89\,143.4\,g+\right.$$
$$\left.1.59603\times 10^{7}\left(-8\,g-6.26556\times 10^{-8}\left(-28\,451.1\,c^2+89\,143.4\,g\right)\right)\right)k\,\mathrm{Sin}\left[\frac{1}{2}\sqrt{54\,403.9\,c^2-k^2}\ t\right]\Bigg)$$
$$(\mathrm{Cos}[10.7992\,x]-\mathrm{Cosh}[10.7992\,x]-1.\,(\mathrm{Sin}[10.7992\,x]-\mathrm{Sinh}[10.7992\,x]))+$$
$$e^{-\frac{k\,t}{2}}\left(\frac{1}{c^2}1.24936\times 10^{-15}\left(2.87302\times 10^{11}\,c^2+173\,889.\,g+\right.\right.$$
$$\left.3.69331\times 10^{8}\left(-8\,g-2.7076\times 10^{-9}\left(-55\,501.1\,c^2+173\,889.\,g\right)\right)\right)\mathrm{Cos}\left[\frac{1}{2}\sqrt{106\,129.\,c^2-k^2}\ t\right]+$$
$$\frac{1}{c^2\sqrt{106\,129.\,c^2-k^2}}1.24936\times 10^{-15}\left(2.87302\times 10^{11}\,c^2+173\,889.\,g+\right.$$
$$\left.\left.3.69331\times 10^{8}\left(-8\,g-2.7076\times 10^{-9}\left(-55\,501.1\,c^2+173\,889.\,g\right)\right)\right)k\,\mathrm{Sin}\left[\frac{1}{2}\sqrt{106\,129.\,c^2-k^2}\ t\right]\right)$$
$$(\mathrm{Cos}[12.7627\,x]-\mathrm{Cosh}[12.7627\,x]-1.\,(\mathrm{Sin}[12.7627\,x]-\mathrm{Sinh}[12.7627\,x]))$$

■ Finding c, k, and g via the Manipulate command - Figures 3, 4, 5, and 6 in the paper are found this way!

Since beam isn't moving until the fourth set of data points, drop the first four sets of data points.

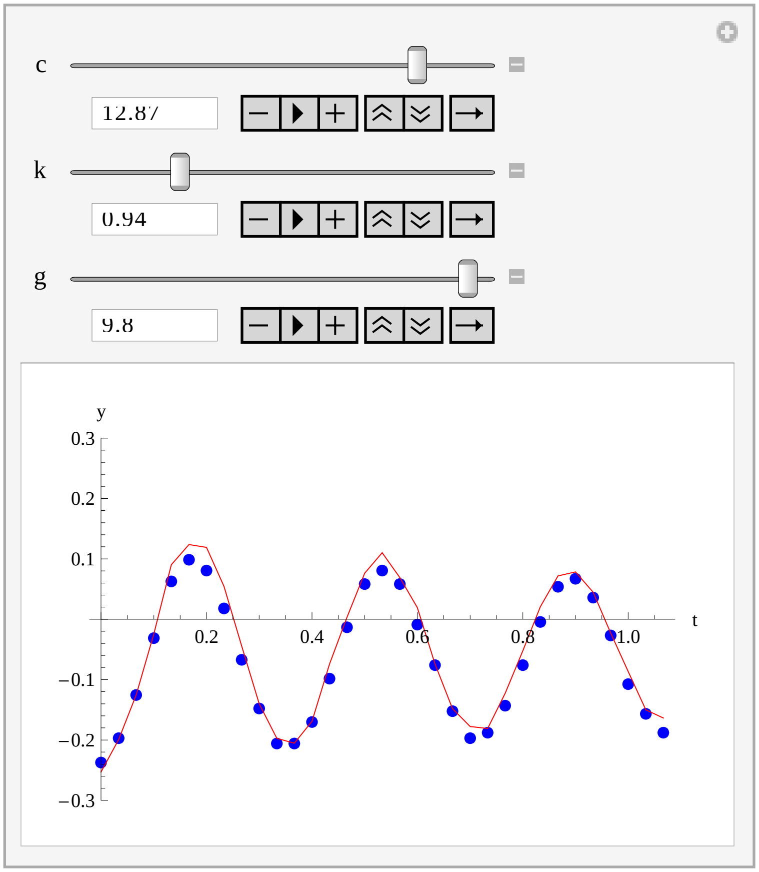
```
In[31]:= Beam = Drop[Beam, {1, 4}];

In[32]:= data = Table[Partition[Beam[[i]], 2], {i, 1, Length[Beam]}];

In[33]:= Clear[c, k, g]
```

In[34]:=
```
Manipulate[Show[ListPlot[Table[{(j - 1)/30, data[[j, 15, 2]]}, {j, 1, Length[Beam]}], PlotStyle →
    {PointSize[0.02], RGBColor[0, 0, 1]}, PlotRange → {-0.3, 0.3}, AxesLabel → {"t", "y"}],
  ListPlot[Table[{(j - 1)/30, Evaluate[y[data[[j, 15, 1]], (j - 1)/30]]}, {j, 1, Length[Beam]}] /. {c → χ1,
     k → χ2, g → χ3}, PlotStyle → RGBColor[1, 0, 0], PlotRange → {-0.3`, 0.3`}, Joined → True]],
 {{χ1, 1, "c"}, 1, 15}, {{χ2, 0, "k"}, 0, 4}, {{χ3, 0, "g"}, 0, 10},
 TrackedSymbols → {χ1, χ2, χ3}, SaveDefinitions → True]
```

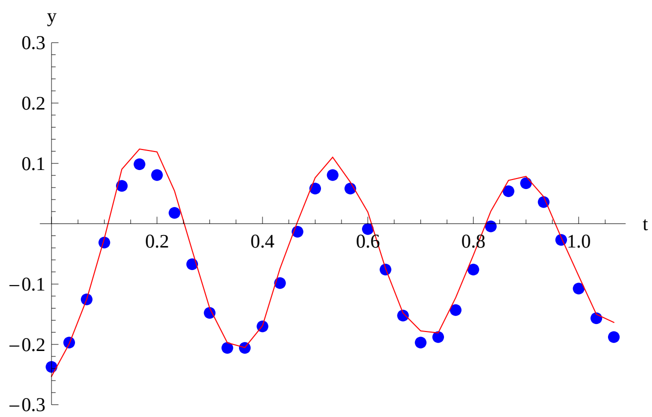
Out[34]=



■ Here's how to plot Figure 6 without using the Manipulate command

In[35]:=
```
Show[ListPlot[Table[{(j - 1)/30, data[[j, 15, 2]]}, {j, 1, Length[Beam]}],
  PlotStyle → {PointSize[0.02], RGBColor[0, 0, 1]},
  PlotRange → {-0.3, 0.3}, AxesLabel → {"t", "y"}],
 ListPlot[Table[{(j - 1)/30, Evaluate[y[data[[j, 15, 1]], (j - 1)/30]]}, {j, 1, Length[Beam]}] /.
   {c → 12.87, k → 0.94, g → 9.8},
  PlotStyle → RGBColor[1, 0, 0], PlotRange → {-0.3`, 0.3`}, Joined → True]]
```

Out[35]=



■ Find error with a choice of c, k, and g via a Mean Sum of the Squares for Error Calculation

Find the error with the choices of c, k, and g (entered as c1, k1, and g1 so that the Manipulate command will work correctly) that give a reasonable fit graphically. The function $\sqrt{H(c, k, g)}$ computes the mean of the sum of the squares for error (MSSE) over all data points.

In[36]:=
```
H[c_, k_, g_] :=
  Evaluate[Sum[(data[[j, i, 2]] - y[data[[j, i, 1]], (j - 1)/30])^2, {i, 1, 15}, {j, 1, Length[Beam]}] /
    (Length[Beam] * 15)]
```

In[37]:=
```
Sqrt[H[12.87, 0.94, 9.8]]
```

Out[37]= 0.0103875

■ Find values of c, k, and g that minimize MSSE via the FindMinimum command, starting with c = c1, k = k1, and fixing g = 9.8 $m/\sec^2$. Then compute error with the resulting c and k!

In[38]:=
```
c1 = 12.87;
k1 = 0.94;
```

In[40]:=
```
FindMinimum[H[c, k, 9.8], {c, c1}, {k, k1}]
```

Out[40]= {0.000105073, {c → 12.842, k → 1.00698}}

In[41]:=
```
√H[12.84, 1.01, 9.8]
```

Out[41]= 0.0102511

- **Plot model vs. measured data at all points with our new choices for c and k with g fixed at 9.8 $m/\sec^2$ - this is Figure 7 in the paper:**

In[42]:=
```
g1a = Show[Table[Graphics3D[{RGBColor[0, 0, 1], PointSize[0.011],
      Point[Table[{xtape⟦j⟧, (i - 1)/30, Beam⟦All, 2 j⟧⟦i⟧}, {i, 1, Length[Beam⟦All, 1⟧]}]]}],
    {j, 1, 15}], Axes → {True, True, True}, AxesLabel → {"x", "t", "y"},
   BoxRatios → {3, 2, 1}, ImageSize → 600];
```

In[43]:=
```
g2a = Show[
   Table[ParametricPlot3D[{xtape⟦i⟧, j/30, y[xtape⟦i⟧, j/30] /. {c → 12.84, k → 1.01, g → 9.8}},
     {j, 0, 32}, Axes → {True, True, True}, AxesLabel → {"x", "t", "y"},
     Ticks → {xtape, Automatic, Automatic}, BoxRatios → {3, 2, 1},
     PlotRange → {{0, L}, {0, Length[Beam] / 30}, {-0.25, 0.25}},
     ImageSize → 600, PlotStyle → {Thick, Red}], {i, 1, 15}]];
```

In[44]:= `Show[g2a, g1a, Boxed → False, ViewPoint → {0.8, -1.5, 1.5}]`

Out[44]=