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L'Hôpital's Rule Via Integration

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In elementary calculus texts L'Hôpital's rule is usually proven only for the case $0/0$, $x \to x_0$ (finite), by applying the Cauchy mean value theorem. Extension to $x \to \infty$ is then accomplished by replacing x with $1/x$. Verification of the rule for the ∞/∞ indeterminate form is regarded as too difficult and may be discussed in an exercise, an appendix, or not at all. In this note we give a proof for the ∞/∞ case that does not make use of the Cauchy mean value theorem. Instead, we require that the functions have continuous derivatives and take advantage of the order properties of the definite integral. The argument adapts nicely to the case $0/0$ as well.

L'HÔPITAL'S RULE. ∞ / ∞. *Let f and g have continuous derivatives with $g'(x) \neq 0$. If* $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} g(x) = \infty$, and $\lim_{x\to\infty} f'(x)/g'(x) = L$, *then* $\lim_{x\to\infty} f(x)/g(x) = L$ *also.*

Proof. We assume that L is finite; the other case can be handled in a similar fashion. The limit hypothesis on g allows us to assume that it is a positive function. Moreover, since g' is continuous and nonvanishing it too must be positive.

Let ε be some positive number. Choose M so that

$$\left|\frac{f'(x)}{g'(x)} - L\right| < \varepsilon$$

whenever $x > M$. Since $g'(x)$ is positive we have

$$|f'(x) - Lg'(x)| < \varepsilon g'(x),$$

so that

$$\left|\int_a^b [f'(x) - Lg'(x)]\,dx\right| \leqslant \int_a^b |f'(x) - Lg'(x)|\,dx < \int_a^b \varepsilon g'(x)\,dx$$

whenever $M < a < b$. Therefore, for such a and b,

$$|f(b) - f(a) - L[g(b) - g(a)]| < \varepsilon[g(b) - g(a)]. \qquad (*)$$

Dividing through by the positive number $g(b)$ we obtain

$$\left|\frac{f(b)}{g(b)} - \frac{f(a)}{g(b)} - L\left[1 - \frac{g(a)}{g(b)}\right]\right| < \varepsilon\left[1 - \frac{g(a)}{g(b)}\right] < \varepsilon.$$

It follows easily that

$$\left|\frac{f(b)}{g(b)} - L\right| < \varepsilon + \frac{|f(a)|}{g(b)} + |L|\frac{g(a)}{g(b)}.$$

As b increases, $g(b)$ grows larger and larger without bound. Consequently, both $|f(a)|/g(b)$ and $|L|g(a)/g(b)$ will eventually become (and remain) smaller than ε, implying that

$$\left|\frac{f(b)}{g(b)} - L\right| < 3\varepsilon$$

for all b sufficiently large. This shows that $\lim_{x\to\infty} f(x)/g(x) = L$. □

Since our proof of this version of L'Hôpital's rule makes no use of the assumption $\lim_{x\to\infty} f(x) = \infty$, that condition can be dropped from the hypotheses. A straightforward variation of the preceding proof works when $x \to x_0$; alternatively, that case can be derived from the $x \to \infty$ case by considering $F(x) = f(x_0 + 1/x)$ and $G(x) = g(x_0 + 1/x)$.

This type of proof also applies to the indeterminate form $0/0$. For example, if we assume that L is finite and g' is a positive function (as was the case above), then allowing b to increase without bound in inequality $(*)$ will force $f(b)$ and $g(b)$ towards 0, implying that

$$|-f(a) - L[-g(a)]| \leqslant \varepsilon[-g(a)].$$

Dividing by the (positive) number $-g(a)$ reveals that

$$\left|\frac{f(a)}{g(a)} - L\right| \leqslant \varepsilon$$

whenever $a > M$.

As you can see, the algebra for $0/0$ is a bit simpler making this proof even more suitable for popular consumption.

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