# Nice Cubic Polynomials, Pythagorean Triples, and the Law of Cosines

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**Problems** This paper arose from an attempt to find examples suitable for a beginning calculus class that illustrate the techniques of sketching graphs of polynomials. An excellent example for this purpose is the cubic polynomial

$$y = 2x^3 + 5x^2 - 4x - 3$$

because both $y = (x+3)(2x+1)(x-1)$ and $y' = 2(x+2)(3x-1)$ factor over the integers. Our attempt to give a systematic description of all such 'nice' cubics led to Diophantine equation (2) below, whose solution is well known. Our techniques give systematic methods for producing nice numbers for other problems as well. Although these problems have been investigated by several authors, [2], [6], [7], [8], [10], [13], the solutions have not been given a systematic and unified presentation. We attempt to do so here, in a way that allows the reader to construct additional examples at will.

The first problem we are interested in is:

> *Given a cubic polynomial, find the three roots, the two critical points, and sketch the graph.*

We want the roots and critical points to have rational $x$-coordinates (with small denominators). If a cubic with rational roots has a double or triple root its derivative will necessarily have rational roots. Thus we assume that the cubic has distinct roots. By multiplication we can assume that the roots are integers, and by translation that the middle root is zero. Thus we are investigating polynomials of the form

$$y = (x+a)x(x-b). \tag{1}$$

We are looking for nonnegative integers $a$ and $b$ so that the derivative $y' = 3x^2 + 2(a-b)x - ab$ has rational roots. This happens when the discriminant is a perfect square. This leads to the Diophantine equation

$$a^2 + b^2 + ab = c^2. \tag{2}$$

If, instead, we make zero the leftmost root, we proceed from

$$y = x(x-a)(x-b)$$

to the equation

$$a^2 + b^2 - ab = c^2. \tag{3}$$

The second problem is the box problem:

> *An open box is constructed from a rectangular piece of metal by cutting four equal squares from the corners and bending up the resulting tabs. Find the dimensions which maximize the volume of the box.*

We want the given dimensions, $a$ and $b$, of the rectangle to be positive integers and the maximum volume to be attained at a rational value for the edge length, $x$, of the squares. In this case we proceed from the equation

$$y = x(a-2x)(b-2x)$$

to equation (3) above.

The third problem reaches back to trigonometry:

> *Using the law of cosines, find the length of the third side of a triangle given the lengths of two sides and the cosine of the included angle $\gamma$.*

We want the lengths of the sides of the triangle, $a$, $b$, and $c$, to be integers. This requires that $\cos\gamma$ be rational. If we let $g = -2\cos\gamma$, this reduces to the equation

$$a^2 + b^2 + gab = c^2. \tag{4}$$

Notice that equations (2) and (3) are the special cases of (4) corresponding to $\gamma = 120°$ and $\gamma = 60°$ respectively, while $g = 0$ gives the familiar Pythagorean equation.

**Solutions** All the solutions are a special case of a general theorem in Dickson [**5**, p. 44], which he gives as an exercise [**5**, p. 48, exercise 6].

If g is rational, $-2 < g < 2$, then all the rational solutions of

$$a^2 + b^2 + gab = c^2$$

are

$$(a,b,c) = k(u^2 - v^2, 2uv + gv^2, u^2 + guv + v^2)$$

where $u$ and $v$ are relatively prime integers, and $k$ is rational.

Let $d = 2 + g$. The assumption $-2 < g < 2$ insures that $d > 0$. We reparameterize by replacing $u$ by $i + j$ and $v$ by $j$ to get

$$(a,b,c) = k(i^2 + 2ij, 2ij + dj^2, i^2 + dij + di^2).$$

This form will be used to give all our solutions.

All cubic polynomials with rational roots and critical points can be obtained by a rational translation along the $x$-axis of a polynomial of the form (1) with

$$(a,b) = k(i^2 + 2ij, 2ij + 3j^2). \tag{5}$$

We refer to such cubics with $k = 1$ as canonical nice cubics.

For the box problem all rational dimensions are given by

$$(a,b) = k(i^2 + 2ij, 2ij + j^2).$$

The maximum occurs at $x = kij/2$ and the dimensions of the box are

$$(ki^2 + kij) \times (kij + kj^2) \times (kij/2).$$

All triangles with three rational sides and a given angle $\gamma$ lying opposite side $c$ are of the form

$$(a, b, c) = k(i^2 + 2ij, 2ij + dj^2, i^2 + dij + dj^2), \tag{6}$$

where $d = 2(1 - \cos\gamma)$. In particular

$$\begin{aligned}
\gamma = 60°: &\quad (a, b, c) = k(i^2 + 2ij, 2ij + j^2, i^2 + ij + j^2)\\
\gamma = 90°: &\quad (a, b, c) = k(i^2 + 2ij, 2ij + 2j^2, i^2 + 2ij + 2j^2)\\
\gamma = 120°: &\quad (a, b, c) = k(i^2 + 2ij, 2ij + 3j^2, i^2 + 3ij + 3j^2).
\end{aligned}$$

**Examples** We first discuss nice cubics. The two simplest canonical cubics are $(x + 3)x(x - 5)$, corresponding to $i = 1$ and $j = 1$ in equation (5), and $(x + 8)x(x - 7)$, corresponding to $i = 2$ and $j = 1$. By negating and translating the roots, we can obtain from each canonical cubic six cubes that have zero as a root. From $(x + 3)x(x - 5)$, we obtain

$$\begin{aligned}
(x + 3)x(x - 5) &= x^3 - 2x^2 - 15x\\
(x + 5)x(x - 3) &= x^3 + 2x^2 - 15x\\
x(x - 3)(x - 8) &= x^3 - 11x^2 + 24x\\
x(x - 5)(x - 8) &= x^3 - 13x^2 + 40x\\
(x + 8)(x + 3)x &= x^3 + 11x^2 + 24x\\
(x + 8)(x + 5)x &= x^3 + 13x^2 + 40x.
\end{aligned}$$

From each of these we can produce nice cubics by translating and multiplying the roots by any rational values. If we use only multiplication, zero is preserved as a root. We multiply by the reciprocals of numbers that divide the product of the nonzero roots. From $x(x - 5)(x - 8)$, for example, we get eight cubics, four of which are

$$\begin{aligned}
x(x - 5)(x - 8) &= x^3 - 13x^2 + 40x\\
x(2x - 5)(x - 4) &= 2x^3 - 13x^2 + 20x\\
x(4x - 5)(x - 2) &= 4x^3 - 13x^2 + 10x\\
x(x - 1)(5x - 8) &= 5x^3 - 13x^2 + 8x.
\end{aligned}$$

The remaining four come from these by interchanging the coefficients of the cubic and linear terms. These with larger leading coefficients have roots bunched around zero and are less suitable for graphing.

The first procedure in this section always yields six cubics from any canonical cubic. The number produced by this second procedure, however, depends on the factorization of the nonzero roots. Only four are produced from $(x + 3)x(x - 5)$ while 16 are produced from $x(x - 8)(x - 15)$, one of the six cubics derived from $(x + 8)x(x - 7)$.

The cubic in the introductory paragraph comes from $(x + 5)x(x - 3)$ by adding $-1$ and then multiplying by $1/2$. Here are four other nice cubics:

$$(x+2)(x-1)(x-6) = x^3 - 5x^2 - 8x + 12$$
$$(x+1)(x-2)(x-7) = x^3 - 8x^2 + 5x + 14$$
$$(x+1)(2x-3)(x-3) = 2x^3 - 7x^2 + 9$$
$$(x+2)(3x-1)(x-3) = 3x^3 - 4x^2 - 17x + 6.$$

For the box problem we insist that $a$ and $b$ be relatively prime integers (but allow fractional box dimensions). This is achieved by letting $k = 1/3$ when 3 divides $i - j$ and $k = 1$ otherwise. It is not difficult to see that if $a$ and $b$ yield a nice box, so do $b - a$ and $b$. Thus rectangle dimensions for boxes come in pairs. The table below gives the five smallest such pairs. They have been ordered by the value of $b$.

| $(i,j)$ | $k$ | $(a,b)$ | $x$ | $(i,j)$ | $k$ | $(a,b)$ | $x$ |
|---|---|---|---|---|---|---|---|
| $(4,1)$ | $1/3$ | $(3,8)$ | $2/3$ | $(2,1)$ | $1$ | $(5,8)$ | $1$ |
| $(3,1)$ | $1$ | $(7,15)$ | $3/2$ | $(5,2)$ | $1/3$ | $(8,15)$ | $5/3$ |
| $(7,1)$ | $1/3$ | $(5,21)$ | $7/6$ | $(3,2)$ | $1$ | $(16,21)$ | $3$ |
| $(5,1)$ | $1$ | $(11,35)$ | $5/2$ | $(7,4)$ | $1/3$ | $(24,35)$ | $14/3$ |
| $(10,1)$ | $1/3$ | $(7,40)$ | $5/3$ | $(4,3)$ | $1$ | $(33,40)$ | $6$ |

These dimensions were published by Duemmel [**6**] who found them by computer search. Graham and Roberts [**10**] have a list of rectangle dimensions that overlaps with these. They cite calculus textbooks that have presented the box problem and observe that only the first three rectangles in the second column have actually been used.

Dundas [**8**] discusses a more practical way of building a box from a rectangular piece of cardboard, with a top and reinforced sides. He shows that this problem involves maximizing the same nice cubic polynomials as the standard box problem discussed here. He also has some interesting discussion about other kinds of boxes.

For triangles here are some triples for $\gamma = 60°$, and $\gamma = 120°$ respectively:

| $(i,j)$ | $(2,1)$ | $(3,1)$ | $(4,1)$ | $(5,1)$ | $(3,2)$ | $(5,2)$ |
|---|---|---|---|---|---|---|
| $(a,b,c)$ | $(8,5,7)$ | $(15,7,13)$ | $(8,3,7)$ | $(35,11,31)$ | $(21,16,19)$ | $(15,8,13)$ |
| $(i,j)$ | $(1,1)$ | $(1,2)$ | $(1,3)$ | $(2,1)$ | $(2,3)$ | $(4,1)$ |
| $(a,b,c)$ | $(3,5,7)$ | $(5,16,19)$ | $(7,33,37)$ | $(8,7,13)$ | $(16,39,49)$ | $(24,11,31)$. |

Finally we look at angles that have a rational cosine. In equation (6) let $m/n$ be $d$ in lowest terms, let $(i,j) = (1,1)$ and multiply by $n$. This produces the triangle $(a,b,c) = (3n, m+2n, 2m+n)$. Here are some of these values:

| $\cos\gamma$ | $3/4$ | $1/4$ | $5/6$ | $2/3$ | $-1/4$ | $-3/4$ |
|---|---|---|---|---|---|---|
| $d$ | $1/2$ | $3/2$ | $1/3$ | $2/3$ | $5/2$ | $7/2$ |
| $(a,b,c)$ | $(6,5,4)$ | $(6,7,8)$ | $(9,7,5)$ | $(9,7,8)$ | $3(2,3,4)$ | $(6,11,16)$. |

**Other possibilities** Obtain the triangle $(a,b,c) = 3(n, n+1, n+2)$ by setting $d = (n+3)/n$ and $\cos\gamma = (n-3)/2n$. The triangle $(a,b,c) = (3n, 3n+1, 3n+2)$ is obtained by setting $d = (n+1)/n$ and $\cos\gamma = (n-1)/2n$. Letting $(i,j) = (1,n)$, yields $(a,b,c) = (1+2n, 2n+mn, 1+m+mn)$. Dundas [**7**] has some interesting observations about triangles with a given rational value for $\cos\gamma$.

**Comments** The result from Dickson given in section 2 has a long history. The case $g = 0$ is the classical problem of finding all Pythagorean triples. This solution appears in Diophantus, Book II, Theorem 8. According to Heath however, it had essentially already appeared in Euclid Book X, Theorem 28, Lemma 1 [**11**, p. 116–7], [**12**, Vol. 3, p. 63–4]. For $g = -1$ this solution was first given by J. Neuberg in 1874–5 [**4**, p. 405, ref. 36], and for $g = 1$ by J. Neuberg and G. B. Mathews in 1887 [**4**, p. 406, ref. 40]. The general theorem was first proved by A. Gérardin in 1911 [**4**, p. 406, ref. 45].

All the sources we have consulted use this form [**5**, p. 42, exercise 1], [**9**, p. 16], [**14**, p. 146]. We prefer our second form. It has the nice property that positive $i$ and $j$ give positive $a$, $b$, and $c$. When $i$ is much bigger, about the same size, or much smaller than $j$, it gives an $a$ that is respectively much bigger, about the same size, or much smaller than $b$. At the start of our investigation we discovered this parameterization even before we had begun to generate triples on the computer.

In our investigation of nice cubics the initial assumption that the middle root be zero is arbitrary. Another approach is to assume a pair of roots of equal magnitude and opposite sign. This is achieved through translation by $(a - b)/2$. The resulting polynomial has the form $(x^2 - r^2)(x - s)$, where

$$r = (a + b)/2 \qquad a = r + s$$
$$s = (a - b)/2 \qquad b = r - s$$

and the condition that it be nice reduces to

$$3r^2 + s^2 = c^2.$$

This transformation eliminates the cross term from the left side of equation (2). This could be achieved by a transformation corresponding to a 45° rotation, which has determinant 1 but irrational entries. The transformation above has rational entries, therefore preserving rational points, but has the disadvantage of a determinant $\neq 1$. Graham and Roberts [**10**] use (essentially) this transformation to solve the box problem. They obtain a substantially different parameterization than the two discussed here.

A linear transformation with determinant 1 that fixes the set of integer solutions to a Diophantine equation is known as an *automorph* [**5**, p. 72–3], [**3**, p. 146]. The proof of the formula for Pythagorean triples given in [**14**, p. 146–7] makes use of the factorization of the left-hand side of equation (4) over the quadratic extension of the rational field obtained by adjoining the square root of the discriminant $g^2 - 4$. When the discriminant is negative, it is known that multiplication by the roots of unity in this extension gives all automorphs.

With just two exceptions, a negative discriminant yields exactly two roots of unity, namely $\pm 1$. The exceptions correspond precisely to the three equations studied here. With $g = 0$ the extension contains the fourth roots of unity. With $g = \pm 1$ the extension contains the sixth roots of unity. This is what lies behind the procedure at the beginning of section 3 in which six nice cubics are generated from each canonical cubic.

A symmetry like interchanging $a$ and $b$ can be very useful even though it has determinant $-1$ and does not qualify as an automorph. We observed that if $a$ and $b$ yield a nice box, so do $b - a$ and $b$. This transformation also has determinant $-1$. It is used in both [**6**] and [**10**].

The relationship of automorphs to the units in a quadratic extension of the rational field is explained in Borevich and Shafarevich [**1**, p. 75–6]. The fact that there are only finitely many units (and therefore automorphs) when the discriminant is negative can be regarded as a very special case of the Dirichlet Unit Theorem [**1**, p. 114].

REFERENCES

1. Z. I. Borevich and I. R. Shafarevich, *Number Theory*, Academic Press, New York, 1966.
2. T. Bruggeman and T. Gush, Nice cubic polynomials for curve sketching, this MAGAZINE 53 (1980), 233–234.
3. H. Davenport, *The Higher Arithmetic*, 5th edition, Cambridge University Press, London, 1982.
4. Leonard E. Dickson, *History of the Theory of Numbers*, Vol. 2, Chelsea Publishing Co., 1952.
5. Leonard E. Dickson, *Introduction to the Theory of Numbers*, Dover Publications, Inc., New York, 1957.
6. James Duemmel, From calculus to number theory, *Amer. Math. Monthly* 96 (1989), 140–143.
7. Kay Dundas, Quasi-Pythagorean triples for an oblique triangle, *The Two-year College Mathematics Journal* 8 (1977), 152–155.
8. Day Dundas, To build a better box, *The College Mathematics Journal* 15 (1984), 30–36.
9. A. O. Gelfond, *The Solution of Equations in Integers*, W. H. Freeman, San Francisco, 1961.
10. *George P. Graham and Charles E. Roberts, A Diophantine equation from calculus, this MAGAZINE 62* (1989), 97–101.
11. Thomas L. Heath, *Diophantus of Alexandria*, Dover Publications, Inc., New York, 1964.
12. Thomas L. Heath, *The Thirteen Books of Euclid's Elements*, Cambridge Univ. Press, London, 1926.
13. O. P. Lossers, Integer roots of cubics, *Amer. Math. Monthly* 96 (1989), 841–842.
14. Harry Pollard and Harold G. Diamond, *The Theory of Algebraic Numbers*, 2nd edition, MAA, 1975.

# A Calculus Exercise For the Sums of Integer Powers

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Let $S_{k,n} = 1^k + 2^k + \cdots + n^k$, where $k, n$ are positive integers. The usual method of finding $S_{k,n}$ for $k = 1, 2, \ldots$ is by means of the identity

$$\sum_{i=0}^{k-1} \binom{k}{i} S_{i,n} = (n+1)^k - 1, \tag{1}$$

where $\binom{k}{i}$ denotes the binomial coefficient $k!/i!(k-i)!$. Recently, D. Acu has obtained (1) and similar formulas involving only even or only odd values of $i$ from certain binomial identities [1].

We shall derive a generalization of (1) by differentiating the function

$$F(x) = e^{ax} + e^{(a+d)x} + e^{(a+2d)x} + \cdots + e^{(a+(n-1)d)x}.$$

Indeed,

$$F^{(k)}(0) = \sum_{m=1}^{n} (a + (m-1)d)^k, \tag{2}$$