and $p(x)$ is a polynomial with $p(x) > 0$ for all x in some interval $(0, d)$. Other families of functions could, of course, also be considered, but this family yields a particularly nice result. The areas we are concerned with are now given by

$$A_T = \frac{1}{2}xf(x) \qquad \text{and} \qquad A_C = \int_0^x f(t)dt$$

and the limit in question by

$$\lim_{x\to 0} \frac{A_T}{A_C} = \lim_{x\to 0} \frac{\frac{1}{2}xf(x)}{\int_0^x f(t)dt}.$$

Using l'Hôpital's rule and the fundamental theorem of calculus, we have

$$\begin{aligned}\lim_{x\to 0} \frac{A_T}{A_C} &= \frac{1}{2} \lim_{x\to 0} \frac{xf'(x) + f(x)}{f(x)} \\ &= \frac{1}{2}\left[\lim_{x\to 0} \frac{xr[p(x)]^{r-1}p'(x)}{[p(x)]^r} + 1\right] \\ &= \frac{1}{2}\left[\lim_{x\to 0} \frac{xrp'(x)}{p(x)} + 1\right].\end{aligned}$$

But $\lim_{x\to 0}[xp'(x)]/[p(x)] = k$, where k is the lowest power that appears in the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_k x^k$ ($a_k \neq 0$). The final result, then, is

$$\lim_{x\to 0} \frac{A_T}{A_C} = \frac{1}{2}(rk + 1).$$

Thus in the special case of the conics, where $r = 1/2$ and $k = 1$,

$$\lim_{x\to 0} \frac{A_T}{A_C} = \frac{1}{2}\left(\frac{1}{2} + 1\right) = \frac{3}{4},$$

the same value obtained for the circle.

A group project that calculus students might find rewarding is first to compute $\lim_{x\to 0} A_T/A_C$ for various specific functions and then attempt to formulate general results.

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Critical Points of Polynomial Families

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Using computers or graphing calculators opens new opportunities for rich classroom investigative experiences. Problems can be stated, not necessarily in general terms, and students are encouraged to experiment, generate patterns, and explore conjectures. The purpose of this note is to share a problem suitable for a first-year calculus course, regarding critical points of a one-parameter family of polynomials $\{f_t(x)\}$.

Since critical points (i.e., points $(u, f(u))$ where $f'(u) = 0$) are used in determining the extrema of a polynomial $f(x)$, it is interesting to see the effect on these points if we allow one of the coefficients of $f(x)$ to vary. The idea was motivated by a laboratory session in a first-year calculus course, where one of us had the students investigate the behavior of quadratic and cubic polynomials. (An exercise of this type appears in Deborah Hughes-Hallett et al., *Calculus,* Wiley, New York, 1994, p. 270, exercise 3.) The following example illustrates our ideas.

Example 1. Consider critical points of the family of polynomials $f_t(x) = tx^4 - 6x^2 + 4$. If $(x, f_t(x))$ is a critical point of f_t, then$f_t'(x) = 0$ implies $4tx^3 = 12x$, so $4tx^4 = 12x^2$. Therefore

$$\begin{aligned} f_t(x) &= tx^4 - 6x^2 + 4 \\ &= \frac{1}{4}(4tx^4 - 24x^2 + 16) \\ &= \frac{1}{4}(12x^2 - 24x^2 + 16) \\ &= \frac{1}{4}(-12x^2 + 16). \end{aligned}$$

Hence each critical point $(x, f_t(x))$ lies on the graph of $p(x) = \frac{1}{4}(-12x^2 + 16)$.

Conversely, suppose that $(x, p(x))$ is an arbitrary point on the graph of $p(x) = \frac{1}{4}(-12x^2 + 16)$. If $x = 0$ then $(x, p(x)) = (0, 4)$ is a critical point of f_t for each t. If $x \neq 0$ then $(x, p(x))$ is a critical point of f_t when $t = 3/x^2$. This is the case, since $f_t(x) = p(x)$ and $f_t'(x) = 4tx^3 - 12x = 0$ when $x \neq 0$ and $t = 3/x^2$. It follows that the locus of critical points $(x, f_t(x))$ of the family is given by the graph of

$$p(x) = \frac{1}{4}(-12x^2 + 16).$$

Figure 1 shows the graphs of the functions $f_t(x) = tx^4 - 6x^2 + 4$ for $t = 1, 2, 3, \ldots, 10$ and the locus of critical points, $p(x) = \frac{1}{4}(-12x^2 + 16)$.

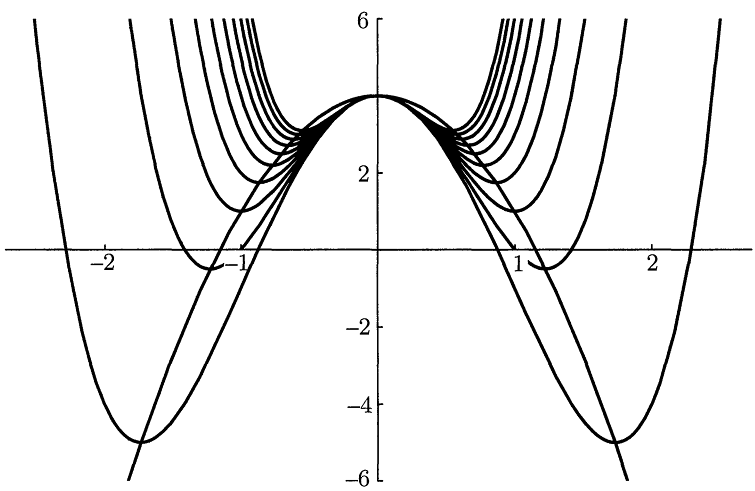

Figure 1

These observations led to the following general result.

Theorem. *Consider the one-parameter family of polynomials* $f_t(x) = g(x) + tx^m$, *where* $m \geq 1$ *and* $g(x)$ *is a nonzero polynomial with no mth-degree term. The locus of critical points* $(x, f_t(x))$ *of the family is given by the graph of*

$$p(x) = g(x) - \frac{g'(x)x}{m} \qquad \text{for } x \neq 0 \tag{1}$$

together with the point $(0, g(0))$ *in case* g *has no first-degree term.*

Proof. Suppose $m = 1$. If $(x, f_t(x))$ is a critical point of $f_t(x) = g(x) + tx$, then

$$f_t'(x) = g'(x) + t = 0.$$

Hence

$$t = -g'(x) \qquad \text{and} \qquad tx = -g'(x)x.$$

Therefore if $(x, f_t(x))$ is a critical point of f_t then

$$f_t(x) = g(x) + tx = g(x) - g'(x)x.$$

Conversely, for every x, the point $(x, g(x) - g'(x)x)$ is a critical point of f_t for $t = -g'(x)$. Therefore the locus of the critical points is given by equation (1) for $m = 1$. Notice that, since g has no first-degree term, $g'(0) = 0$ and $(0, g(0))$ is a critical point of f_0. Hence the point $(0, g(0))$ is included in the locus of critical points $(x, f_t(x))$ of the family $\{f_t\}$.

Suppose that $m \geq 2$. If $(x, f_t(x))$ is a critical point of $f_t(x) = g(x) + tx^m$, then $f_t'(x) = g'(x) + tmx^{m-1} = 0$. Hence

$$tmx^{m-1} = -g'(x)$$

and

$$tx^m = -\frac{g'(x)x}{m}.$$

Therefore if $(x, f_t(x))$ is a critical point of f_t, then

$$f_t(x) = g(x) + tx^m = g(x) - \frac{g'(x)x}{m}.$$

Conversely, suppose we consider an arbitrary point $(x, g(x) - \frac{1}{m}g'(x)x)$. If $x = 0$ and g has no first-degree term, then $g'(0) = 0$ and $(x, g(x) - \frac{1}{m}g'(x)x) = (0, g(0))$ is a critical point of f_t for all t. If $x = 0$ and g has a first-degree term, then $g'(0) \neq 0$ and $(0, g(0))$ is not a critical point of any f_t. But if $x \neq 0$, then $(x, g(x) - \frac{1}{m}g'(x)x)$ is a critical point of f_t when $t = -g'(x)/(mx^{m-1})$. This completes the proof of the theorem. ■

We have two remarks. First, if $m = 0$ in the theorem, then $f_t(x) = g(x) + t$ where g is a polynomial with no constant term; so the graph of $f_t(x)$ is just a vertical translate of the graph of $g(x)$. The locus of the critical points $(x, f_t(x))$ of the family $\{f_t\}$ therefore consists of vertical lines passing through the critical points of g. Secondly, if the degree of $f_t(x)$ is n, then the locus of critical points $p(x)$ is a polynomial with $\deg(p(x)) = n = \deg(g(x))$ if $m \neq n$, and $\deg(p(x)) = \deg(g(x)) < n$ if $m = n$.

The following examples illustrate the theorem.

Example 2. If $f_t(x) = x^4 + tx^3 - 6x^2 + 4$, then $g(x) = x^4 - 6x^2 + 4$ and the locus of critical points $(x, f_t(x))$ is given by the graph of the equation $p(x) = g(x) - \frac{1}{3}g'(x)x = \frac{1}{3}(-x^4 - 6x^2 + 12)$. Figure 2 shows the graphs of the functions $f_t(x) = x^4 + tx^3 - 6x^2 + 4$ for $t = 0, 1, 2, \ldots, 10$ and the locus of critical points, $p(x) = \frac{1}{3}(-x^4 - 6x^2 + 12)$.

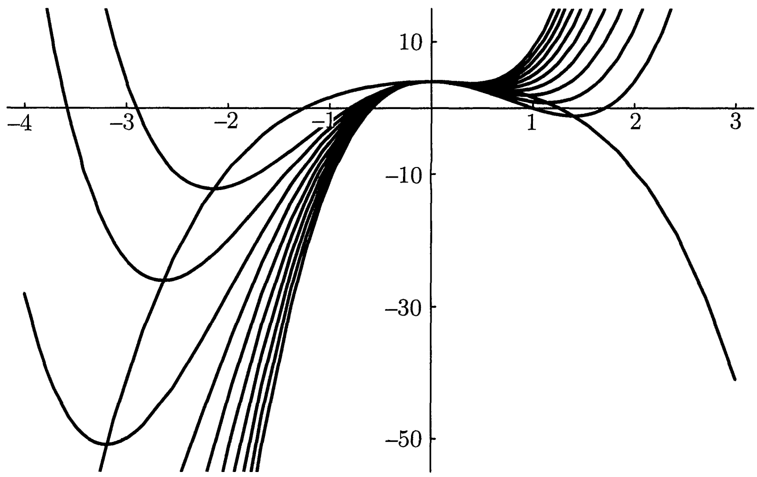

Figure 2

Example 3. If $f_t(x) = x^4 + tx^2 + 4$, then $g(x) = x^4 + 4$ and the locus of critical points $(x, f_t(x))$ is given by the graph of the equation $p(x) = g(x) - \frac{1}{2}g'(x)x = \frac{1}{2}(-2x^4 + 8)$. Figure 3 shows the graphs of $f_t(x) = x^4 + tx^2 + 4$ for $t = -1, -2, -3, \ldots, -10$ and the locus of critical points, $p(x) = \frac{1}{2}(-2x^4 + 8)$.

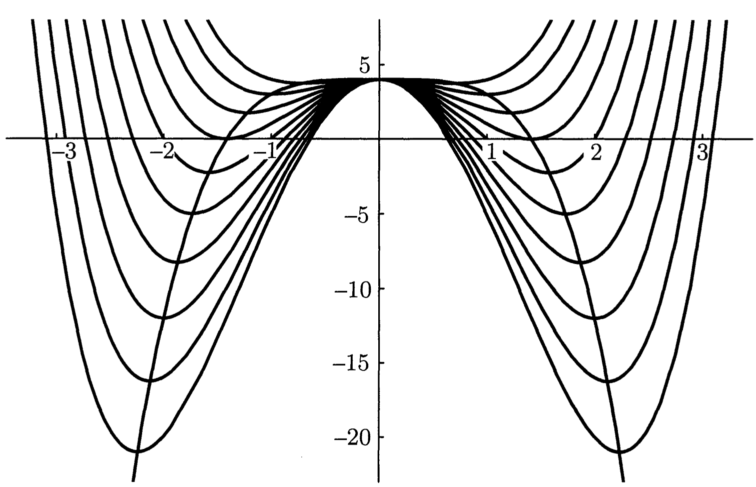

Figure 3

Example 4. If $f_t(x) = 2x^3 + tx^2 + x + 3$, then $g(x) = 2x^3 + x + 3$ and the locus of critical points $(x, f_t(x))$ is given by the graph of the equation $p(x) = g(x) - \frac{1}{2}g'(x)x = \frac{1}{2}(-2x^3 + x + 6)$ for $x \neq 0$. Notice that when $x = 0$ the point $(x, g(x) - \frac{1}{2}g'(x)x) = (0, 3)$ is not a critical point of any f_t. Figure 4 shows the graphs of $f_t(x) = 2x^3 + tx^2 + x + 3$ for $t = -6, -5, -4, \ldots, 4, 5, 6$ and the locus of critical points, $p(x) = \frac{1}{2}(-2x^3 + x + 6)$ for $x \neq 0$.

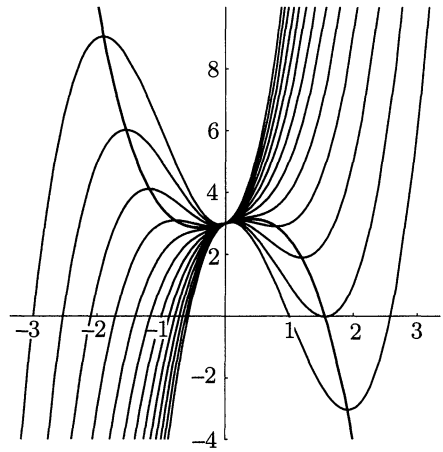

Figure 4

Editor's note. The main results in this capsule appeared in an article by F. Alexander Norman [Parameter-Generated Loci of Critical Points of Polynomials, *CMJ* 19:3 (1988) 223–229], which includes references to the research literature.

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How to Fix a Broken Heart

In 1784 [Muzio] Clementi [1752–1832; composer] returned to the Continent and eloped with an eighteen-year-old girl he had met on his previous trip, the daughter of a prosperous Lyons merchant. The enraged father pursued the couple, and with the aid of authorities soon reclaimed his daughter. Heartbroken, Clementi retired to Bern, where he consoled himself by working at mathematics. By late 1784 or early 1785 he felt sufficiently restored to return to London, but lingering melancholy apparently kept him from performing.

William H. Youngren, "Finished Symphonies,"
Atlantic Monthly 277:5 (May 1996), page 105.
Contributed by Ed Barbeau, University of Toronto.