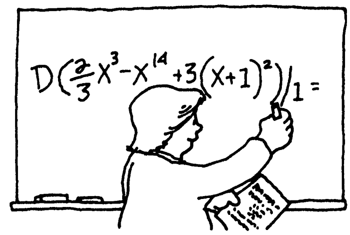
CLASSROOM CAPSULES

EDITOR

Thomas A. Farmer
Department of Mathematics and Statistics
Miami University
Oxford, OH 45056-1641



A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Tom Farmer.

Cylinder and Cone Cutting

Michael R. Cullen (mcullen@popmail.lmu.edu), Loyola Marymount University, Los Angeles, CA 90045-2699

It is well known that when you tilt an ordinary cup of water, the shape of the water's surface is an ellipse. If the cup is cylindrical, we can give an analytic proof by rotating the cylinder $x^2 + y^2 = r^2$ through an angle θ in the yz-plane. Using the rotation of axes formulas, we obtain its new equation

$$x^2 + (y\cos\theta + z\sin\theta)^2 = r^2$$

and then observe (Figure 1) that the plane $z = 0$ cuts this rotated cylinder in the standard ellipse

$$x^2 + (\cos^2\theta)y^2 = r^2.$$

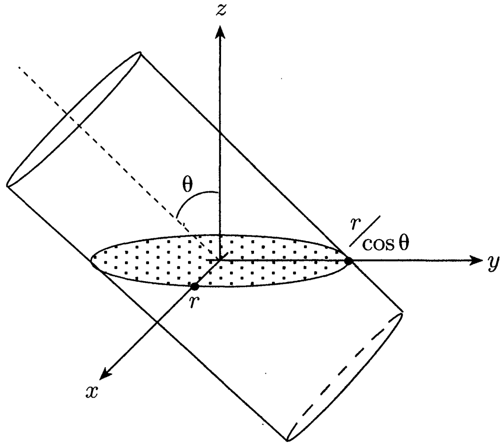


Figure 1

Students are aware that conic sections are formed when a cone is cut by various planes. It is difficult, however, to provide an elementary demonstration of this fundamental result that does not involve some subtle geometry in three dimensions. The following demonstration is a simple argument based on *rotating the cone* through various angles while keeping the plane fixed. It has the advantage that the equations of the sections are the standard equations that students encounter in a precalculus course.

When the standard cone $z^2 = x^2 + y^2$ is cut by the plane $z = r$, we obtain the circle $x^2 + y^2 = r^2$. Rather than cutting this cone by other planes, let's rotate the cone through an angle θ $(0 \le \theta \le \pi/2)$ about the x-axis (Figure 2). The equation of the rotated cone is

$$(-y \sin\theta + z\cos\theta)^2 = x^2 + (y\cos\theta + z\sin\theta)^2.$$

Now let $z = r$ to obtain the equation

$$(-y \sin\theta + r\cos\theta)^2 = x^2 + (y\cos\theta + r\sin\theta)^2 \tag{1}$$

for the curve. By expanding and using the double angle identities, (1) can be simplified to

$$x^2 + (\cos 2\theta)y^2 + (2r\sin 2\theta)y = r^2\cos 2\theta \tag{2}$$

and completing the square gives (provided $\cos 2\theta \neq 0$)

$$\frac{x^2}{\cos 2\theta} + \frac{(y + r\tan 2\theta)^2}{1} = r^2 \sec^2 2\theta,$$

which is a standard equation of a translated conic. The nature of the conic section is completely determined by the sign of $\cos 2\theta$. When $\cos 2\theta > 0$ $(0 < \theta < \pi/4)$, the conic is an ellipse. When $\cos 2\theta = 0$ $(\theta = \pi/4)$, equation (2) reduces to $x^2 + 2ry = 0$, the equation of a parabola. Finally, when $\cos 2\theta < 0$ $(\pi/4 < \theta \le \pi/2)$, we obtain a hyperbola. Note that in the special case $\theta = \pi/2$, the equation reduces to the standard hyperbola $y^2 - x^2 = r^2$.

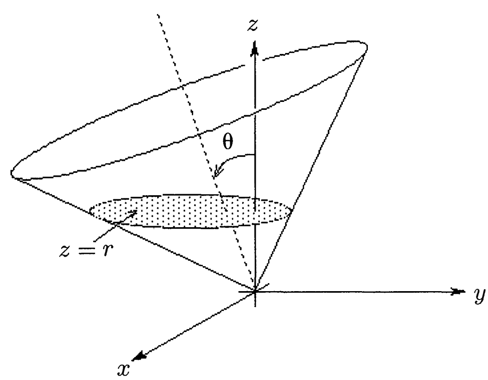


Figure 2

———— o ————