It follows that

$$\frac{(n+1)^{(n^3-3n-2)/3n^3}\cdot e^{[1-(n+1)^3]/9n^3}}{n^{1/3}} < \frac{\left[1^{(1^2)}\cdot 2^{(2^2)}\cdots n^{(n^2)}\right]^{1/n^3}}{n^{1/3}} < \frac{(n+1)^{(n^3+3n^2+3n+1)/3n^3}\cdot e^{[1-(n+1)^3]/9n^3}}{n^{1/3}},$$

or

$$\frac{(n+1)^{1/3}}{n^{1/3}}\cdot(n+1)^{(-3n-2)/3n^3}\cdot e^{[1-(n+1)^3]/9n^3} < \frac{\left[1^{(1^2)}\cdot 2^{(2^2)}\cdots n^{(n^2)}\right]^{1/n^3}}{n^{1/3}} < \frac{(n+1)^{1/3}}{n^{1/3}}\cdot(n+1)^{(3n^2+3n+1)/3n^3}\cdot e^{[1-(n+1)^3]/9n^3}.$$

Now, note that as $n \to \infty$:

$$\frac{(n+1)^{1/3}}{n^{1/3}} \to 1, \qquad (n+1)^{(-3n-2)/3n^3} \to 1,$$

$$(n+1)^{(3n^2+3n+1)/3n^3} \to 1, \qquad e^{[1-(n+1)^3]/9n^3} \to e^{-1/9}.$$

Hence,

$$\lim_{n\to\infty} \frac{\left(1^{(1^2)}\cdot 2^{(2^2)}\cdots n^{(n^2)}\right)^{1/n^3}}{n^{1/3}} = e^{-1/9}.$$

In the same manner, we can begin with

$$f(x) = \frac{x^{s+1}\ln x}{s+1} - \frac{x^{s+1}}{(s+1)^2}$$

to establish (2) in general.

Notational Collisions

J. Hillel, Concordia University, Montreal, Quebec, Canada

"Interpreting a symbol is to associate it with some concept or mental image to assimilate it to human consciousness."

The Mathematical Experience, Davis and Hersh

Mathematics is often eulogized for its compact and concise notational systems and for its use of symbols to represent (possibly quite complex) objects, constructions, and operations. Yet we know that for learners of mathematics, the attempt to attach meaning to symbols and notations is at the heart of a multitude of cognitive difficulties. For example, there is the well-known difficulty in sorting out whether a letter signifies a variable, an unknown, a constant ("a variable at rest" [Bourbaki]), a parameter ("a constant that varies"), etc.

Notational collisions refers to a more mundane, yet common situation, which also seems to be the cause of many cognitive obstacles; namely, when a letter is used to signify one thing and afterwards it is used to signify something else. A frequent case of this occurs in teaching, when the use of symbols within a general theory (global use) clashes with their use in special cases of the theory (local use) where the choice of symbols is determined by some historical precedent or by an appeal to more familiar conventions. The intent of this capsule is to illustrate a few examples of notational collisions from linear algebra.

Consider the notational collision between the representation of a system of m equations in n unknowns as

$$\begin{gathered} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m, \end{gathered}$$

and the more familiar convention for 2×2 (or 3×3) systems as in

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2. \end{aligned}$$

This may not seem to represent much of a shift, but think of the student who is trying to make sense out of the matrix equation $A\vec{x} = \vec{b}$. In the general case, A, $\vec{x}$ and $\vec{b}$ match faithfully with the letters used for their respective components. But in this specific case, $\vec{b}$ is the vector $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ rather than $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, and b_1, b_2 are now the y-coefficients in the two equations.

Another illustration of notational collisions is the use of A^T to denote the transpose of a matrix A, after the letter T has been used consistently to represent a linear transformation. This is just plain sloppiness. But what about the frequent use of A^t and A' as other standard notations for the transpose of a matrix? The former clashes head-on with the notation for powers of a matrix, and this creates some interesting anomalies such as having the coexisting relations $(A^t)^t = A$ and $(A^m)^n = A^{mn}$. (Should one need to specify $m \neq t$ and $n \neq t$ in the second relation?) The notation A' for the transpose of A collides with the derivative notation (as, for example, the "derived" matrix whose elements are the derivatives of the original matrice's elements) or with the common use of A and A' to designate two arbitrary elements (as in the relation $\operatorname{tr}(A + A') = \operatorname{tr} A + \operatorname{tr} A'$).

Mathematics teachers know that symbols convey only the meaning that we attribute to them, and that we are empowered to change either the symbols or their meaning as we see fit (though some symbols, such as $\int$, are so deeply entrenched in the mathematical culture that they have the status of 'untouchables'). We are also fairly adept at adjusting to changes in notations or symbols (perhaps not as adept as we would like to be—as a student, I used to have great difficulty with the first editions of Van der Waerden's *Modern Algebra* simply because he used capital gothic letters to denote the elements of a set).

However, good pedagogical sense certainly favors consistent use of symbols and the avoidance of whimsical changes. The implicit message we pass on to our students is that, within a given course, specific symbols are the carriers of specific concepts, and that it is unnecessary to make these concepts explicit for every use of the symbols. This message is often communicated by the gradual relaxation of details that we give to symbols and notation. Thus, at the beginning of a linear algebra course, we may say "the system of equations $A\vec{x}=\vec{b}$, where

$$A=\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \qquad \vec{x}=\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \qquad \text{and} \qquad \vec{b}=\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}."$$

But eventually we talk about "an $m \times n$ system $A\vec{x}=\vec{b}$ with A as the coefficient matrix." And finally, we may simply say "consider $A\vec{x}=\vec{b}$." At this point, we have sealed an unwritten contract with our students about the shared interpretation of the symbols A, $\vec{x}$, and $\vec{b}$.

Students, in fact, take symbols much too literally. They have trouble dealing with changes in labelling, as when using w instead of x for a variable, or using B instead of A to represent some arbitrary matrix. By contrast, they seem to derive a strong sense of security from a consistent use of symbols, since these symbols become associated for them with particular concepts (P and Q with nonsingular matrices, X and Y with column vectors, etc.). Since students' established symbol-concept pairings have an important mnemonic function and are usually consistent with our implicit teaching strategy, notational collisions create feelings of deception: students perceive them as an arbitrary and unilateral break of the implicit 'contract.'

A notational collision 'deluxe'. My concern with notational collisions stems from a particular experience in teaching an introductory linear algebra course. Just at what should have been the highlight of such a course, the text contained a spectacular collision (to be described below) in which *all* the global symbols of the general theory clashed with the local symbols of the special application. Since I tended to adhere to the text's notation in an effort not to confuse the students, I simply used the same symbols in my lectures.

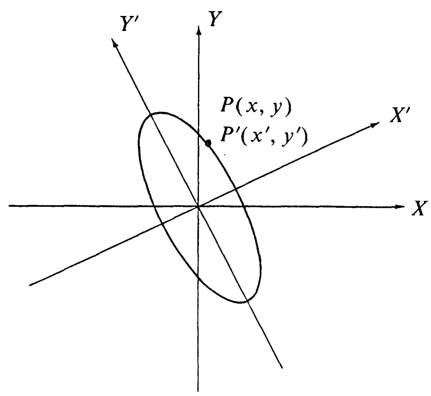
Now, it is fairly traditional to conclude a linear algebra course with some results on diagonalization of matrices and, in particular, with the principal axis theorem that establishes the existence of an orthogonal basis of eigenvectors for any real symmetric matrix. This theorem has a beautiful geometric application: the classification of (nondegenerate) equations of the second degree in 2 and 3 variables into conics and quadric surfaces (or, as we often tell the students, "eliminating the mixed terms" from a second degree equation).

This material is usually covered in a rather short span of time at the end of the course. Although it is really just about the existence of nontrivial solutions of the system $(A-\lambda I)\vec{x}=\vec{0}$ and about the rank of such a system, we seem to forget how many concepts we introduced along the way (eigenvalue, eigenvector, characteristic polynomial, similarity, congruence, quadratic form, etc.). I once counted up to 35 new definitions related to the topic of diagonalization. My point here is that some notational collisions come at a particularly inopportune time, when the students are already at the 'overload' point.

The text I was using for the course was fairly standard in its notation. However, as the diagonalization theory is developed, some of the symbols used acquire a more specialized meaning than they had in the general context:

A, B	matrices, now always square and even symmetric.
P, Q	nonsingular matrices, now often orthogonal.
X, Y	column vectors.
X', A'	transposed matrices.
$B = P'AP$	congruence relation.

In the text's section on 'simplifying' the general equation of the second degree in two variables, such an equation is written as $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Then, to illustrate what is happening geometrically, the text has the following figure:



Most of us would not find anything unusual in all this; both the global and local use of the symbols are familiar and, more or less, standard. After the end of the lecture on the second degree equation, one of the better students in the class admitted that he was confused. The dialogue between us went something like this:

"I don't understand the meaning of Ax^2 here (pointing to the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$)."
"Why is this a problem?"
"Well, A is a matrix and x^2 is a number, so what does Ax^2 mean?"
"Hold it, A is not a matrix, it is just the coefficient of x^2."
"*I thought that A was always a matrix in this course.*"

Only when attempting to 'straighten out' this student did I become aware of the multiple collisions created by the text and myself:

A, B	are no longer matrices but coefficients of terms in the second degree equation.
P	is no longer an orthogonal matrix but a point in the plane.
X, Y	are no longer column vectors but labels for the usual coordinate system.
X', Y'	are not transposes but the new coordinate system.

Quite a mess!

Notational collisions raise several questions, not the least of which is "what's this fuss all about?" We can simply take the attitude that the ability to interpret a symbol appropriately in a given context is part and parcel of doing mathematics; that it is up to the students themselves to make the necessary separation between symbol and meaning. There is, of course, a ring of truth to this but that doesn't exempt us from being more judicious in our selection of symbols.

The above "deluxe" example illustrates a collision between two standard notations which suddenly had to co-exist. With a bit of foresight, most of the confusion could have been eliminated. Since such examples are rampant in our courses, we should keep notational collisions in mind the next time the perennial "why can't they understand?" question comes up amongst colleagues.

A Closer Look

Self-similarity is an easily recognizable quality. Its images are everywhere in the culture: in the infinitely deep reflection of a person standing between two mirrors, or in the cartoon notion of a fish eating a smaller fish eating a smaller fish eating a smaller fish. Mandelbrot likes to quote Jonathan Swift:

"So, Nat'ralists observe, a Flea
Hath smaller Fleas that on him prey,
And these have smaller Fleas to bite'em,
And so proceed ad infinitum."

James Gleick, *Chaos*: *Making a New Science*, Penguin Books

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The largest island in a lake is Manitoulin Island (1,068 square miles) in the Canadian (Ontario) section of Lake Huron. The island itself has a lake of 41.09 square miles on it, called Manitou Lake, which is the world's largest lake within a lake, and in that lake are a number of islands.

Guiness Book of World Records 1987, Bantam Books